ALGEBRA QUALIFYING EXAMINATION

AUGUST 2017

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A. Let $N \in \mathrm{Mat}_n(\mathbb{C})$ be an $n \times n$ matrix. Assume that N is nilpotent; that is, there exists an integer $r > 0$ such that $N^r = 0$.

(a) Prove that if N is diagonalizable, then $N = 0$.

(b) Prove that $I_n + N$, where I_n denotes the $n \times n$ identity matrix, is invertible.

1B. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$ be linear transformations of real vector spaces.

(a) Prove that BA is never invertible.

(b) Give an example showing that AB can be invertible.

(c) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric linear transformation of real vector spaces. Suppose that trace(T^2)=0. Prove that $T = 0$.

2A. Let $G = \mathrm{GL}_3(\mathbb{F}_7)$ be the group of invertible 3×3 matrices over the finite field $\mathbb{F}_7$ with 7-elements. Find, with proof, a 7-Sylow subgroup of G.

2B.

(a) If $|G| = p^n q$ with $p > q$ primes, prove that G contains a unique normal subgroup of index q.

(b) Suppose G is a finite p-group and $H \neq \{1\}$ is a normal subgroup of G. Show that $H \cap Z(G) \neq 1$, where $Z(G)$ is the center of G.

3A. Let A be a commutative ring with unit.

(a) Let $a \in A$. Show that

$$\mathrm{Ann}(a) = \{b \in A \,:\, ba = 0\}$$

is an ideal.

(b) Assume that A is in addition noetherian and $1 \neq 0$. Prove that there exists $a \in A$ such that the ideal $\mathrm{Ann}(a)$ is prime in A.

3B.

(a) Prove that a non-zero prime ideal in a PID is maximal.

(b) Find, with proof, a maximal ideal I in $\mathbb{Z}[\sqrt{-1}] = \{m + n\sqrt{-1} | m, n \in \mathbb{Z}\}$ which contains 5, given the fact that $\mathbb{Z}[\sqrt{-1}]$ is a PID (in fact, it is also Euclidean).

4A. Let K be a splitting field for the polynomial $X^5 - 7$ over $\mathbb{Q}$. Find, with proof, $[K : \mathbb{Q}]$ and an explicit Galois field extension $\mathbb{Q} \subsetneq L \subsetneq K$ with $[L : K] = 2$.

4B. Let K be the splitting field of $x^3 - 11 \in \mathbb{Q}[x]$ over $\mathbb{Q}$. Find, with proof, the degree $[K : \mathbb{Q}]$ and the Galois group $\mathrm{Gal}(K/\mathbb{Q})$. Determine explicitly all the intermediate fields between $\mathbb{Q}$ and K.

5A. Let A be a noetherian commutative ring with unit. Let M be a finitely generated A-module. Let $\phi \colon M \to M$ be an A-module homomorphism. Prove that if ϕ is surjective, then it is an isomorphism (this is even true when A is non-noetherian).

5B. Suppose R is a commutative ring and M is an R-module. An R-submodule N of M is called pure if $rN = N \cap rM$ for all $r \in R$.

(a) Show that any direct summand of M is pure.

(b) Assume that R is an integral domain. If M/N is torsion-free show that N is pure. Prove the converse when M is in addition assumed to be torsion-free.