ALGEBRA QUALIFYING EXAMINATION

JANUARY 2018

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A. Let X be an $n \times n$ matrix over a field k such that $X^2 = X$.

(1) Prove that X is diagonalizable over k.
(2) If Y is another $n \times n$ matrix over k such that $Y^2 = Y$, then Y is conjugate to X if and only if $\operatorname{rank} X = \operatorname{rank} Y$.

1B. Let $T : V \to V$ be a linear operator on a finite-dimensional inner product space V over a field k and T^* be its adjoint. Prove that T is an orthogonal projection if and only if $T^2 = T = T^*$.

2A. Let $p < q$ be primes.

(1) Assume that $p \nmid q - 1$. Prove that every group of order pq is cyclic.
(2) Give an explicit example of primes $p < q$, where $p \mid q - 1$ and a finite group of order pq that is not cyclic.

2B.

(1) Show that there is no simple group of order 200 $(= 2^3 \cdot 5^2)$.
(2) Prove that any group of order 588 $(= 2^2 \cdot 3 \cdot 7^2)$ is solvable, given that any group of order 12 is solvable.

3A. Let A be a commutative noetherian ring with 1.

(1) Let $I \subseteq A$ be an ideal. Prove that A/I is a noetherian ring.
(2) Give an explicit example of a noetherian ring A and a unital subring $B \subseteq A$ such that B is not a noetherian ring.

3B. Let $D = \mathbb{Z}[\sqrt{5}] = \{m + n\sqrt{5} | m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{5})$, the field of fractions of D. Show the following:

(1) $x^2 + x - 1$ is irreducible in $D[x]$.
(2) $x^2 + x - 1$ is not irreducible in $F[x]$.
(3) D is not a unique factorization domain.

4A. Let $f(x) = (x^2 - 3)(x^2 - 5)(x^2 - 15)$. Calculate the splitting field K of f over $\mathbb{Q}$, the group $\mathrm{Gal}(K/\mathbb{Q})$, and list all fields L such that $\mathbb{Q} \subseteq L \subseteq K$.

4B. Let $f(x) = x^4 + x^3 + x^2 + x + 1$. Let ω be a root of the polynomial $f(x)$ in $\mathbb{C}$ and $K = \mathbb{Q}(\omega)$. Prove that $K/\mathbb{Q}$ is a Galois extension, determine the Galois group $\mathrm{Gal}(K/\mathbb{Q})$, and list all fields L such that $\mathbb{Q} \subseteq L \subseteq K$.

5A. Let A be a commutative ring with 1. Let M and N be finitely generated A-modules. Prove that $M \otimes_A N$ is a finitely generated A-module.

5B. Let M be a finitely generated R-module and $\mathfrak{a} \subset R$ an ideal in a commutative ring R with 1. Suppose $\psi : M \to M$ is an R-module homomorphism such that $\psi(M) \subset \mathfrak{a}M$. Find a monic polynomial $p(t) \in R[t]$ with coefficients from $\mathfrak{a}$ such that $p(\psi) = 0$.