ANALYSIS QUALIFYING EXAM
AUGUST, 2014

Please show all your work

GOOD LUCK!

Problem 1

Let $g(x)$ be a real valued function on $[0, \infty)$ which is C^1 and suppose that $g'(x)$ is bounded. Suppose also that $g(0) = 0$. Find (with proof)

$$\lim_{n \to \infty} n \int_0^\infty \frac{g(x/n)}{x} e^{-x}\, dx.$$

Problem 2

Find all positive values of α for which the formula

$$A_\alpha u(x) = \int_0^1 \frac{u(y)}{(x+y)^\alpha} dy$$

defines a bounded operator in $L^1([0,1])$. Compute its norm.

Problem 3

Let $f(x)$ be a differentiable, real valued function on $[0,1]$. Suppose that

$$\int_0^1 \frac{|f'(x)|^3}{x} dx = 1.$$

Prove that

$$f(1) - f(0) \leq \left(\frac{2}{3}\right)^{2/3}. \tag{1}$$

Find all functions $f(x)$ for which (1) becomes an equality.

Problem 4

Prove that the series

$$\sum_{n=1}^\infty \sin\left(\frac{x}{n^2}\right)$$

converges for every x and its sum is a continuous function on $(-\infty, \infty)$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Problem 5

Find all real values of α for which the function $x\sin(x^{-\alpha})$ is absolutely continuous on the interval $(0,1)$.

Problem 6

Recall that a metric space is separable if there is a countable dense subset in it. The set of finite signed Borel measures on the real line is a metric space if we define the norm $||\mu|| = |\mu|(\mathbb{R})$ where $|\mu|$ is the total variation of the measure μ.
(a) Prove that the space of finite signed Borel measures on $\mathbb{R}$ that are absolutely continuous with respect to Lebesgue measure m is a separable metric space. You may use, without proving it, the fact that the space $L^1(\mathbb{R}, m)$ is separable.
(b) Prove that the space of all finite signed Borel measures on $\mathbb{R}$ is not separable.