GEOMETRY/TOPOLOGY QUALIFYING EXAM
JANUARY 2018

Please show all your work. GOOD LUCK!

Problem 1

Find a conformal mapping from the strip $\{z : 0 < \text{Im} z < 1\}$ to the first quadrant $\{\zeta : \text{Re}\zeta > 0,\ \text{Im}\zeta > 0\}$ of the complex plane.

Problem 2

Let X and Y be manifolds, and let $C_0^\infty(X)$ and $C_0^\infty(Y)$ be spaces of smooth, compactly supported functions on X and Y, respectively. Let $p : X \to Y$ be a smooth covering. For $f \in C_0^\infty(X)$, the function $p_* f$ on Y is defined by the formula

$$p_* f(y) = \sum_{x \in p^{-1}(y)} f(x).$$

Prove that p_* is a surjective map from $C_0^\infty(X)$ to $C_0^\infty(Y)$.

Problem 3

Let

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

be a vector field in $\mathbb{R}^2$. What condition a differential form $\omega = a(x, y)dx \wedge dy$ must satisfy for $\mathcal{L}_X \omega = 0$? Here $\mathcal{L}_X$ is the Lie derivative. Suppose that $a(x, y)$ is not identically equal to 0. Can it be smooth? Give an example of a smooth, non-zero two-form ω in $\mathbb{R}^2 \setminus \{0\}$ such that $\mathcal{L}_X \omega = 0$.

Problem 4

Let $GL_+(2, \mathbb{R})$ be the space of all 2×2 matrices with real entries the determinant of which is positive. Compute $\pi_1(GL_+(2, \mathbb{R}))$.

Problem 5

Let $a_1, \cdots, a_k$ be k distinct points on the two-dimensional sphere S^2, and let $X = S^2 \setminus \{a_1, \dots, a_k\}$. Compute singular homologies of X with integer coefficients.

Problem 6

Compute $\pi_1(\mathbb{R}P^2 \times S^1)$. Here $\mathbb{R}P^2$ is the space of all lines passing through the origin in $\mathbb{R}^3$, and S^1 is the unit circle in $\mathbb{R}^2$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX