LINEAR ALGEBRA PROBLEMS

UNIVERSITY OF ARIZONA INTEGRATION WORKSHOP, AUGUST 2015

1. Lecture 1

Problem 1.1. Prove that matrix multiplication is associative by relating this operation to composition of linear maps (via choices of bases).

Problem 1.2. Let V be a vector space and S a basis of V.

(1) Prove that $S^* \subseteq V^*$ is a linearly independent set.
(2) If V is of finite dimension, prove that S^* is a basis of V^*.
(3) Give an example to show that S^* need not span V^* in general.

Problem 1.3. Let U_1, U_2 be subspaces of a finite dimensional vector space V. Prove that

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

Problem 1.4. Prove or disprove: For U_1, U_2, U_3 subspaces of a finite dimensional vector space V,

$$\begin{aligned}\dim(U_1 + U_2 + U_3) &= \dim(U_1) + \dim(U_2) + \dim(U_3) \\ &\quad - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Problem 1.5. A system of linear equations over an infinite field may have no solutions, a unique solution, or infinitely many solutions. Explain this behavior in terms of kernels of linear transformations.

Problem 1.6. Let V, W be vector spaces of dimensions n and m, respectively. Show that $\mathrm{Hom}(V, W)$ is naturally a vector space and that it has dimension mn.

Problem 1.7. Let $\{W_i\}_i$ be a collection of vector spaces over a field k, indexed by some set I, and set $V := \bigoplus_i W_i$. Express V^* in terms of the W_i^*. Do not assume I is finite, nor that the W_i are of finite dimension.

Problem 1.8. Let V be a finite dimensional vector space and $T \in \mathrm{Hom}(V, V)$ a linear map.

(1) Show that T is nilpotent if and only if there is a basis of V in which the matrix of T is upper triangular with zeroes on the diagonal. Deduce that if T is nilpotent, than $T^{\dim(V)} = 0$.
(2) Assuming that k is algebraically closed, show that T is semisimple if and only if there is a basis of V in which the matrix of T is diagonal.
(3) Give an example of V and T which is neither nilpotent nor semisimple.

Problem 1.9. Let V be a vector space of dimension n over a field k, and let $T \in \mathrm{Hom}(V, V)$.

(1) Prove that there is a basis of V in which the matrix of T is upper triangular if and only if there is a chain of T-stable subspaces
$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$
with $\dim(V_i) = i$.
(2) If $k = \mathbf{C}$, then prove that T has an eigenvector. Hint: for any nonzero $v \in V$, the $n + 1$ vectors $v, Tv, \ldots, T^n v$ are linearly dependent.
(3) Using (1) and (2), deduce that when $k = \mathbf{C}$, there is a basis of V in which the matrix of T is upper-triangular.

Problem 1.10. Let $V := \mathbf{F}_3[x]/(x^{12})$, viewed as a vector space over the field $\mathbf{F}_3$ with 3 elements. Let $T : V \to V$ be the linear operator given by $(d/dx)^2$.

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These problems borrow heavily from versions from previous years, especially those written by Nick Rogers, Dinesh Thakur, and Doug Ulmer, as well as old University of Arizona Algebra Qualifying Exams.

(1) Find the matrix of T with respect to the basis $\{x^i\}_{0\leq i\leq 11}$.
(2) Find bases for the kernel and image of T.

Problem 1.11. Let $T : V \to W$ be a linear transformation and $T^\vee : W^* \to V^*$ the dual (transpose) map. Prove that $\ker(T) = \mathrm{im}(T^\vee)^\perp$ where for a subspace $U \subset V^*$ we set

$$U^\perp := \{v \in V : u(v) = 0 \text{ for all } u \in U\}$$

Similarly, prove that $\mathrm{im}(T) = \ker(T^\vee)^\perp$.

Problem 1.12. If $W \subseteq V$ is a subspace of a finite dimensional vector space with $\dim(W) = \dim(V) - 1$, prove that there exists $f \in V^*$ with $W = \ker(f)$.

Problem 1.13. Suppose given a system of linear equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad i = 1, \ldots, m. \tag{1.1}$$

with $a_{ij}, b_i \in \mathbf{R}$ for all i, j. If (1.1) has a solution in $\mathbf{C}^n$, show that it has a solution in $\mathbf{R}^n$. In this case, are all solutions in $\mathbf{R}^n$?

Problem 1.14. Using Gaussian elimination, show that any matrix over a field can be decomposed as a product PLU with P a square permutation matrix, L a square lower-triangular matrix, and U a (non necessarily square) upper triangular matrix.

Problem 1.15. Let k be a finite field of size q.
(1) Show that the number of $n \times n$ matrixes over k is q^{n^2}.
(2) Find (and prove!) a formula for the number of invertible $n \times n$ matrices over k.
(3) (Challenge) Find and prove a formula for the number of nilpotent $n \times n$ matrices over k.

2. Lecture 2

In what follows, unless stated to the contrary all vector spaces are finite dimensional over $\mathbf{C}$.

Problem 2.1. Give necessary and sufficient conditions for an endomorphism $T : V \to V$ to have a square root.

Problem 2.2. Prove that any $n \times n$ matrix over $\mathbf{C}$ is conjugate to its transpose.

Problem 2.3. Let A and B be $n \times n$ matrices. Prove that AB and BA have the same characteristic polynomial.

Problem 2.4. The order n Vandermonde determinant is

$$V_n := \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}$$

Prove that $(x_j - x_i)$ divides V_n for all $i < j$ and conclude that

$$V_n = \prod_{1\leq i<j\leq n} (x_j - x_i).$$

Problem 2.5. Prove that the set of diagonalizable $n \times n$ matrices in $\mathbf{C}^{n^2}$ is dense in the following sense: Given $A \in \mathrm{Mat}_{n\times n}(\mathbf{C})$ and $\varepsilon > 0$, there exists $B \in \mathrm{Mat}_{n\times n}(\mathbf{C})$ all of whose entries are $< \varepsilon$ with $A + B$ diagonalizable. Hint: first prove this for Jordan blocks.

Problem 2.6. If T is an endomorphism of a vector space V with $T^m = \mathrm{id}$, prove that T is diagonalizable. Using your proof, find a more general criterion for diagonalizability.

Problem 2.7. Prove that a real 2×2 matrix with positive off-diagonal entries is diagonalizable.

Problem 2.8. Give an example of a finite dimensional vector space V over $\mathbf{F}_p$ and an endomorphism T satisfying $T^m = \mathrm{id}$ for some m which is *not* diagonalizable.

Problem 2.9. Let $T : V \to V$ be the linear map of problem 1.10. Determine the Jordan canonical form of T.

Problem 2.10. Let A and B be $n \times n$ matrices over a field k, and assume that $AB - BA = B$.

(1) If the characteristic of k is zero or is larger than n, prove that B is nilpotent.
(2) If $\mathrm{char}(k) = p$ and $p \leq n$, is B necessarily nilpotent?

Problem 2.11. Let $V := \mathrm{Mat}_{n\times n}(\mathbf{C})$ and for a fixed $n \times n$ intertible matrix A, denote by $T_A : V \to V$ the linear map determined by $T_A(B) = ABA^{-1}$.

(1) If A is diagonalizable, prove that T_A is too. (**Hint:** Reduce to the case of diagonal A.)
(2) Is T_A diagonalizable for general invertible A?

Problem 2.12. For $T \in \mathrm{Hom}(V, V)$ and $v \in V$ the *cyclic subspace* $C(v)$ generated by v is $\mathrm{span}(\{T^i(v)\}_{i\geq 0})$.

(1) Show that $\dim(C(v)) = 1$ if and only if v is an eigenvector.
(2) More generally, relate $\dim(C(v))$ (for various v) to the Jordan form of T.

Problem 2.13 (Uniqueness of abstract Jordan form)**.** Let $T \in \mathrm{Hom}(V, V)$ have (abstract) Jordan decomposition $T = S + N = S' + N'$ with S, S' semisimple and N, N' nilpotent.

(1) Prove that S and N can be taken to be polynomials in T.
(2) Show that $SS' = S'S$ and $NN' = N'N$.
(3) Show that $S - S'$ is semisimple and $N - N'$ is nilpotent.
(4) Conclude that $S = S'$ and $N = N'$.

Problem 2.14. Determine the set of matrices which commute with a given matrix in each of the following cases:

(1) A is diagonal with distinct eigenvalues.
(2) A is diagonal with not necessarily distinct eigenvalues.
(3) A is a Jordan block.
(4) A is in Jordan canonical form.
(5) General A.

Problem 2.15. For $n \geq 0$, let F_n be the n-th Fibonacci number, defined recursively by $F_0 := 0$, $F_1 := 1$, and $F_n := F_{n-1} + F_{n-2}$. Find an explicit formula for the n-th Fibonacci number.

3. Lecture 3

Problem 3.1. Let V be a finite dimensional vector space over a field k, and f a nondegenerate, symmetric bilinear form on V. Prove that $\dim(W) + \dim(W^\perp) = \dim V$ for any k-subspace W of V.

Problem 3.2. Using the Gram-Schmidt algorithm, prove that every invertible complex matrix can be written as a product QR with Q unitary and R upper-triangular. State and prove the version of this theorem over the real numbers.

Problem 3.3. Prove that any invertible real matrix A can be written as a product QP with Q orthogonal and P symmetric and positive definite (called the polar decomposition of A). Hint: Consider A^tA, which is symmetric and positive definite, and take its square root. State and prove the version of this theorem over $\mathbf{C}$.

Problem 3.4. Let V be a vector space over a field k and $f : V \times V \to k$ a bilinear form. Prove that f induces a nondegenerate form

$$f : V/\ker(L_f) \times V/\ker(R_f) \to k$$

and conclude that $V/\ker(L_f) \simeq (V/\ker(R_f))^*$ and hence that $\mathrm{im}(L_f) \simeq \mathrm{im}(R_f)^*$.

Problem 3.5. The *trace* of an $n \times n$ matrix A, denoted $\mathrm{Tr}(A)$, is the sum of the diagonal entries of A. Equivalently, it is the coefficient of x^{n-1} in $\mathrm{char}_A(x)$. Using the fact that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, show that the pairing on $\mathrm{Mat}_{n\times n}(\mathbf{R})$ given by $\langle A, B\rangle := \mathrm{Tr}(AB)$ is a symmetric bilinear form. Determine its signature.

Problem 3.6. The following statement is false: if $T : V \to V$ is an endomorphism with V of finite dimension over $\mathbf{C}$, then there exists a positive definite Hermitian form on V with respect to which T is normal.

(1) Explain why this statement is false.
(2) Add an hypothesis which makes the conclusion true.
(3) What happens if we change "normal" to "unitary" or to "Hermitian"?

Problem 3.7. Let P_n denote the real vector space of polynomials in one variable x over $\mathbf{R}$ of degree $\leq n$. Show that the pairing

$$\langle , \rangle : P_n \times P_n \to \mathbf{R} \qquad \text{given by} \qquad \langle f, g\rangle := \frac{d^n}{dx^n}(fg)\bigg|_{x=0}$$

is a symmetric bilinear form, and compute its signature.

Problem 3.8. Let P_2 be the space of real polynomials in x of degree ≤ 2, equipped with the inner product

$$\langle f, g\rangle := \int_0^1 fg\, dx.$$

Find an orthonormal basis of P_2.

Problem 3.9. Prove that the set of unitary $n \times n$ matrices is a compact subset of $\mathbf{C}^{n^2}$.

Problem 3.10. Let k be a field and V a vector space over k. For a nondegenerate bilinear form $f : V \times V \to V$and a linear map $A : V \to V$, the *adjoint of A with respect to f* is defined the composite

$$V \xrightarrow{R_f} V^* \xrightarrow{A^\vee} V^* \xrightarrow{R_f^{-1}} V\ .$$

We denote the adjoint of A with respect to f by A^* (and leave f implicit).

(1) Prove that A^* is the unique linear map satisfying $f(Aw, v) = f(w, A^*v)$ for all v and w in V.
(2) How does the notion of adjoint change if we use L_f in place of R_f in the definition?
(3) Classify all nondegenerate pairings $f : V \times V \to V$ on the vector space $V := k^n$ with the property that for *all* linear maps $A : V \to V$, the matrix of A^* in the standard basis of V is the transpose of the matrix of A in the standard basis of V.

Problem 3.11. Suppose that A is a real, symmetric matrix with the property that every negative eigenvalue of A has even multiplicity. Prove that A has a real square root B.

Problem 3.12. Let

$$A := \begin{pmatrix} 1 & 4 & -2 \\ 4 & 1 & -2 \\ -2 & -2 & -2 \end{pmatrix}.$$

Find an orthogonal matrix P so that $P^{-1}AP$ is diagonal.

Problem 3.13. Let A be an $n \times n$ real skew-symmetric matrix.

(1) Prove that the nonzero eigenvalues of A are purely imaginary.
(2) Prove that $\det(A + I_n) \geq 1$.

Problem 3.14. Let V be the space of continuous real-valued and functions on the interval $[-\pi, \pi]$, equipped with the bilinear form

$$\langle f, g\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$$

and let W be the subspace of V spanned by the the collection of functions $B := \{\frac{1}{\sqrt{2}}, \cos(kx), \sin(kx)\}_{k\geq 1}$.

(1) Prove that $\langle\cdot,\cdot\rangle$ restricts to an inner product on W, and that B is an orthonormal basis of W.
(2) Is $\langle\cdot,\cdot\rangle$ an inner product on V? If so, can you describe the orthogonal complement of W?

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