Analysis Problems

Integration Workshop 2017, Department of Mathematics, University of Arizona

Compiled by Ibrahim Fatkullin
ibrahim@math.arizona.edu

Basic concepts, sequences, convergence

1. Every bounded increasing sequence in $\mathbb{R}$ is Cauchy (do not use completeness of $\mathbb{R}$). Every bounded sequence in $\mathbb{R}^n$ has a converging subsequence.

2. A metric space is compact if and only if it is complete and totally bounded, i.e., for every $r > 0$, it may be covered by a finite number of open balls of radius r.

3. Every open set in $\mathbb{R}$ is a union of at most a countable number of disjoint open intervals.

4. Compute $\lim_{n\to\infty} \sqrt[n]{n}$ without explicitly using that $(\ln n)/n \to 0$ as $n \to \infty$.

Series

5. Prove that e is irrational. *(Hint: estimate approximation errors for partial sums of a series representation of* e *or some appropriate quantity related to it.)*

6. A conditionally convergent series (i.e., a non-absolutely convergent series whose partial sums still converge) may be summed to any desired number by an appropriate *rearrangement* of its terms.

7. A sequence is called *Cesàro summable* if the arithmetic means of its partial sums converge. What is the value of the sum $1 - 1 + 1 - 1 + \cdots$ in Cesàro sense?

8. Find examples of converging and diverging series for which $\lim_{n\to\infty} |x_{n+1}/x_n| = 1$ and $\lim_{n\to\infty} \sqrt[n]{|x_n|} = 1$.

9. If the coefficients of a power series are integers, infinitely many of which are nonzero (i.e., the series is not a polynomial) then the radius of convergence of this series is at most 1.

10. Prove that the radii of convergence of power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$ are the same.

11. Suppose all $x_n \geq 0$ and the series $\sum_{n=0}^{\infty} x_n$ converges. Set $y_n = \sum_{m=n}^{\infty} x_m$. Prove that $\sum_{n=0}^{\infty} x_n/y_n$ diverges, while $\sum_{n=0}^{\infty} x_n/\sqrt{y_n}$ converges.

12. Prove that $\sum_{n=1}^{\infty} x_n$ converges iff $\prod_{n=1}^{\infty}(1 + x_n)$ converges.

Continuity

13. Prove that a map of a metric space into a metric space is continuous iff pre-image of every open set is open. Show that if we replaced "pre-image" by "image," the statement would be wrong.

14. An image of a compact set under a continuous map is compact.

15. **Intermediate value theorem.** An image of a connected set under a continuous map is connected. (A set $\mathcal{A}$ is called *connected* if it cannot be *minimally* covered by two disjoint open sets.)

16. Level sets of continuous functions are closed.

17. A function continuous on a compact set $\mathcal{A}$ is also uniformly continuous on $\mathcal{A}$.

18. A function $f : \mathcal{X} \to \mathbb{R}$ is called convex if inequality,

$$f\big(\alpha x_1 + (1-\alpha)x_2\big) \le \alpha f(x_1) + (1-\alpha) f(x_2)$$

holds for all $x_1, x_2 \in \mathcal{X}$ and $\alpha \in [0,1]$. Prove that convex functions are continuous.

19. A function $f : \mathcal{X} \to \mathcal{Y}$ is called **Hölder continuous** with exponent $\alpha \in [0,1]$ if there exists some constant C such that for all $x_1, x_2 \in \mathcal{X}$,

$$d_{\mathcal{Y}}\big(f(x_1), f(x_2)\big) \le C d_{\mathcal{X}}^{\alpha}(x_1, x_2).$$

Prove that if $\alpha > 0$, f is continuous; if $\alpha > 1$, f is constant. (Hölder continuity with $\alpha = 1$ is also referred to as **Lipschitz** continuity.)

20. Construct a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous on all irrationals and discontinuous on all rationals. Prove that the opposite is impossible.

Differentiability

21. Is there a function, differentiable on all irrationals and discontinuous on all rationals?

22. Suppose some *sublevel set*, $\mathcal{F} = \{x : f(x) \le F\}$, of a differentiable function $f : \mathcal{X} \to \mathbb{R}$ is compact, then f achieves its minimum at some $x \in \mathcal{F}$, and its derivative at x vanishes.

23. Prove Taylor's theorem using the mean value theorem.

24. If partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$ are bounded in a neighborhood of x, then f is continuous at x.

25. Find a function discontinuous at the origin whose partial derivatives at the origin are nevertheless well-defined.

26. If there exists a function $\mathbf{D}f(x_0) : \mathcal{X} \to \mathcal{Y}$, such that for all $x \in \mathcal{X}$,

$$\lim_{\epsilon \to 0} \frac{\| f(x_0 + \epsilon x) - f(x_0) - \epsilon\, \mathbf{D}f(x_0; x) \|}{\epsilon} = 0,$$

it is called the **directional (Gâteaux) derivative** of f at x_0. Give examples of non-differentiable functions which are Gâteaux-differentiable. *(Hint: this may happen if, e.g.,* $\mathbf{D}f(x_0; x)$ *is not a linear map of x.)* Suppose $\mathbf{D}f(x_0)$ exists and is linear, would this imply Fréchet differentiability as well?

27. Give example of a function whose derivative at 0 is equal to 1, though the function itself is not invertible in any neighborhood of 0.

Integration

28. A function is of bounded variation iff it may be represented as a difference of two monotone-increasing functions.

29. **Integral test for convergence of series.** Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is monotone-decreasing, then

$$\sum_{n=1}^{\infty} f(n) \quad \text{converges iff} \quad \int_1^{\infty} f(x)\,\mathrm{d}x \quad \text{converges.}$$

30. Prove that if $\int_0^1 f(x)x^n\,\mathrm{d}x = 0$ for all $n = 0, 1, 2\ldots$ and f is continuous, then $f \equiv 0$ on $[0, 1]$.

31. Show by direct computation that

$$\int_1^{\infty}\left(\int_1^{\infty} \frac{x^2 - y^2}{(x^2 + y^2)^2}\,\mathrm{d}y\right)\mathrm{d}x = -\int_1^{\infty}\left(\int_1^{\infty} \frac{x^2 - y^2}{(x^2 + y^2)^2}\,\mathrm{d}x\right)\mathrm{d}y = \frac{\pi}{4}.$$

32. Let Ω be an open bounded subset of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Prove that

$$\mathrm{Vol}(\Omega) = \iint_{\Omega} \mathrm{d}x\,\mathrm{d}y = \oint_{\partial\Omega} x\,\mathrm{d}y = -\oint_{\partial\Omega} y\,\mathrm{d}x = \frac{1}{2}\oint_{\partial\Omega}\left[x\,\mathrm{d}y - y\,\mathrm{d}x\right].$$

Sequences of functions

33. Partial sums of power series and their derivatives (of all orders) converge uniformly on compact subsets of their open intervals of convergence.

34. For real-valued functions on a metric space $\mathcal{X}$, define the *supremum norm:*

$$\|f\| = \sup_{x \in \mathcal{X}} |f(x)|.$$

The set of all continuous functions for which $\|f\| < \infty$ is called $\mathsf{C}(\mathcal{X})$. When is $\mathsf{C}(\mathcal{X})$ a complete metric space with respect to the metric $d(f, g) = \|f - g\|$?

35. Suppose $\{f_n(x)\}$ is a sequence of differentiable functions converging uniformly to $f(x)$. Give an example illustrating that $f(x)$ need not be differentiable. Give an example illustrating that the derivatives $f_n'(x)$ need not converge. Suppose that $f(x)$ is differentiable and $f_n'(x)$ converge point-wise, show that the equality $\lim_{n\to\infty} f_n'(x) = f'(x)$ need not hold.

36. **Peano's existence theorem.** Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous in a neighborhood of (x_0, y_0). Then there exists a function $y(x)$, such that $y(x_0) = y_0$ and $y'(x) = f\big(x, y(x)\big)$. *(Hint: construct Euler approximations to the solution of this differential equation and show that they constitute an equicontinuous family of functions.)*