Integration Workshop 2003
Project on Compact Hausdorff Spaces and C^* algebras

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Part I

Definition: A C^*-algebra U is a complex algebra equipped with a norm $\|\cdot\|$, with respect to which U is complete as a topological space, together with a bijective conjugate linear mapping $*\colon U \to U$, called the adjoint, such that for each $A, B \in U$ and $\lambda \in \mathbb{C}$,

$$(A^*)^* = A,\ (A+B)^* = A^* + B^*,\ (AB)^* = B^*A^*,\ (\lambda A)^* = \bar{\lambda}A^*,$$

$$\|AB\| \leq \|A\|\,\|B\|,\ \text{and the } C^*\text{-}identity,\ \|A^*A\| = \|A\|^2, \text{holds.}$$

Let X be a compact Hausdorff topological space and let $C(X)$ denote the continuous complex valued functions on X. These next few problems prove that $C(X)$ is a commutative unital (i.e. it has a multiplicative identity) C^* algebra with the $*$ operation given by complex conjugation.

1. Show that $C(X)$ is a complex vector space under pointwise addition, i.e. $(f+g)(x) = f(x) + g(x)$, and in fact is a $\mathbb{C}$-algebra when equipped with pointwise multiplication.
2. Show that the map $f \mapsto \bar{f}$, where $\bar{f}$ denotes the complex conjugate of f, is an isometry with respect to the max norm on $C(X)$,
$$\|f\|_\infty = \max_{x \in X} |f(x)|.$$
3. Prove that $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$.
4. Prove that $C(X)$ is complete with respect to the metric induced from the norm $\|\cdot\|_\infty$.
5. Let A and B be unital C^* algebras. A $*$-homomorphism $\phi\colon A \to B$ is a $\mathbb{C}$-algebra homomorphism that commutes with the operation of conjugation, i.e.
$$\phi(a^*) = \phi(a)^*$$

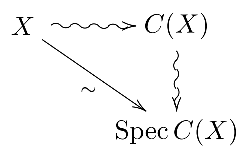
for each $a \in A$. Let $\phi\colon C(X) \to \mathbb{C}$ be a non-zero $*$-homomorphism and show that ϕ is given by evaluation at some $x \in X$, i.e. there exists $x \in X$ such that

$$\phi(f) = f(x)$$

for each $f \in C(X)$. Prove this by contradiction as follows: Suppose $\phi\colon C(X) \to \mathbb{C}$ is a non-zero $\mathbb{C}$-algebra homomorphism, but $\phi \neq \mathrm{eval}_x$ for all $x \in X$.

(a) For each $x \in X$, find $g_x \in C(X)$ such that $g_x \in \ker\phi$, but $g_x(x) \neq 0$.
(b) Show that $|g_x|^2$ is in the ideal $\ker\phi$.
(c) Use the compactness of X to show that the kernel of ϕ is $C(X)$.

Hence, we have a $1-1$ correspondence between the points of X and the nonzero $\mathbb{C}$-algebra homomorphisms $C(X) \to \mathbb{C}$.



Our goal now is to show that this is much more than a correspondence – it is a homeomorphism of compact Hausdorff spaces. In order to do this, we must put a topology on $\operatorname{Spec} C(X)$. The set $\operatorname{Spec} C(X)$ is actually a subset of a much larger space.

Let A be a commutative unital C^*-algebra. The *dual space* of A, denoted $A^\vee$, is the set of continuous complex linear maps $f\colon A \to \mathbb{C}$. Clearly, $\operatorname{Spec} A \subset A^\vee$. There is a topology on $A^\vee$, called the *weak-* topology*. This topology has a basis of open sets, each depending on 3 parameters, ϕ, ϵ and S. The basis is the collection of open sets

$$\mathcal{N}(\phi : S, \epsilon) := \left\{ \omega \in A^\vee \;\middle|\; |\omega(a) - \phi(a)| < \epsilon \;\forall\, a \in S \right\}.$$

Here, $\phi \in A^\vee$, $\epsilon > 0$, and S is a finite subset of A. Hence, $G \subseteq A^\vee$ is open if and only if for each $a \in G$ there exists $\mathcal{N}(b : S, \epsilon)$, an element of the basis, such that $a \in \mathcal{N}(b : S, \epsilon) \subseteq G$.

7. Prove that this topology on $A^\vee$ is Hausdorff.

Consider any $\omega \in A^\vee$. Because ω is linear, $\omega(0) = 0$. Also, ω is continuous, so there exists an open neighborhood U of $0 \in A$ such that $|\omega(a)| \leq 1$ for every $a \in U$. Because U is an open set in the metric space A, there exists $\delta > 0$ such that the ball of radius δ centered at 0 is contained in U.

Now, for every $a \in A$,

$$\left\| \frac{\delta a}{2\|a\|} \right\| < \delta.$$

Hence

$$\left|\omega\left(\frac{\delta a}{2\|a\|}\right)\right| \leq 1.$$

Therefore,

$$\frac{|\omega(a)|}{\|a\|} \leq \frac{2}{\delta}.$$

So there is a bound for $\frac{|\omega a|}{\|a\|}$, independent of $a \in A$. Define

$$\|\omega\| := \sup_{a \neq 0} \frac{|\omega(a)|}{\|a\|}$$

to be the best possible bound.

8. Show this is a norm on $A^\vee$.

9. Show that $\operatorname{Spec} A$ is closed. *Hint: Prove the complement is open in the weak-* topology. Notice if $\phi \notin \operatorname{Spec} A$, then $\phi = 0$ or there exist $a, b \in A$ such that $\phi(a)\phi(b) \neq \phi(ab)$. In the second case, show that $\mathcal{N}(\phi : \{a, b, ab\}, \epsilon)$ lies in the complement of* $\operatorname{Spec} A$ *for some $\epsilon > 0$.*

10. Show that $\operatorname{Spec} C(X)$ is contained in the ball of radius 1 in $C(X)^\vee$.

It is a theorem of Banach and Alaoglu that the unit ball in $A^\vee$ is compact in the weak-* topology.

11. Argue that $\operatorname{Spec} C(X)$ is a compact Hausdorff space.

Define $\Phi\colon \operatorname{Spec} C(X) \to X$ by $\Phi(\mathrm{eval}_x) = x$. To show that Φ is continuous we will need the following Lemma.

Theorem 1 (Uryshon's Lemma) *Let Z be a compact Hausdorff space, and let $z \in Z$ and $Y \subset Z$ be closed, with $z \notin Y$. Then there exists a continuous function $f\colon Z \to [0,1]$ such that $f(z) = 1$ and $f(y) = 0$ for all $y \in Y$.*

12. Show that Φ is continuous. *Hint: Let C be a closed set in X. For any $y \notin C$, consider the neighborhood $\mathcal{N}\left(\mathrm{eval}_y : \{f_y\}, \frac{1}{2}\right)$, where f_y should be inspired from Uryshon's Lemma.*

13. Prove that a continuous bijection between compact Hausdorff spaces is a homeomorphism. What can you conclude?

Part II

We now want to prove a similar statement for C^*-algebras, namely that A is *-isomorphic to $C(\operatorname{Spec} A)$ for any commutative unital C^*-algebra A. The *Gelfand transform* of $x \in A$ is given by

$$\hat{x} : \operatorname{Spec} A \to \mathbb{C} \qquad \hat{x}(\ell) := \ell(x), \quad \ell \in \operatorname{Spec} A.$$

The next series of exercises establishes that the Gelfand transform is an isometric *-isomorphism of C^*-algebras.

Let $A^\times$ be the set of invertible elements of A. For any $x \in A$, the *spectrum* of x is the set

$$\sigma(x) := \{\lambda \in \mathbb{C} | \lambda \cdot 1_A - x \notin A^\times\}.$$

Here, 1_A denotes the identity element of A.

14. Fix $x \in A$. Prove that the range of $\hat{x}$ is $\sigma(x)$.

Notice that we can consider $\hat{}$ as a map $A \to C(\operatorname{Spec} A)$.

15. Show that $\hat{}$ is a homomorphism of $\mathbb{C}$-algebras. We will show that it is in fact a *-homomorphism in what follows.

16. Prove that $\|\hat{x}\|_\infty \leq \|x\|$ for each $x \in A$.

This proves that $\hat{}$ is a continuous map, because it is linear.

One can show, analogous to the classical Spectral Theorem in linear algebra, that for self-adjoint $x \in A$ (that is, $x^* = x$), one has that $\sigma(x) \subseteq \mathbb{R}$. The proof involves a little more functional analysis than we can present here, so we omit.

17. Show that $\hat{}: A \to C(\operatorname{Spec} A)$ is a *-homomorphism, that is $\overline{\hat{x}} = \widehat{x^*}$. *Hint: Notice that if $x = x^*$, then $\hat{x}$ is real valued. Decompose x as $a + ib$, where a is self-adjoint, and b is skew-adjoint, meaning $b = -b^*$.*

The *spectral radius* of $x \in A$ is

$$r(x) := \sup_{\lambda \in \sigma(x)} |\lambda|.$$

One can show that

$$\lim_{n \to \infty} \|x^n\|^{\frac{1}{n}}$$

always exists and is equal to $r(x)$.

18. Use this formula to show that if x is self-adjoint, then $r(x) = \|x\|$. *Hint: Consider x^{2^n}.*

19. Prove that if x is self-adjoint, then $\|\hat{x}\|_\infty = \|x\|$.

20. Use the previous exercise to prove that for every $x \in A$, $\|\hat{x}\| = \|x\|$, that is, $\hat{}$ is an isometry. *Hint: Use the C^*-identity, and recall that x^*x is self-adjoint.*

21. Argue that $\hat{}$ is injective. *Hint: This follows immediately from the previous exercise and the fact that $\hat{}$ is a homomorphism.*

The last remaining obstacle to proving Gelfand's theorem is the surjectivity of $\hat{}$. To do this, we will use the Stone-Weierstrass theorem.

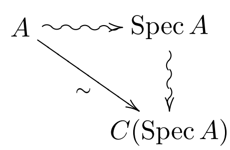
Theorem 2 (Stone-Weierstrass) *Let Z be a compact Hausdorff space, and let B be a closed *-subalgebra of continuous functions on Z that separate points of Z, and contains the constant function, then $B = C(Z)$.*

22. Prove that $\hat{A} := \{\hat{x} | x \in A\}$ separates the points of $\operatorname{Spec} A$.

From this, we see that $\hat{}$ is surjective if its range is closed.

23. The standard proof that the range of $\hat{}$ uses nets and so is a little beyond the scope of the project. Hints for another proof may be available during the workshop.

This completes the theorem of Gelfand, giving the correspondence for commutative unital C^*-algebras:



Part III

24. Consider the unit interval $[0, 1]$ as a subset of $\mathbb{R}$ with the subspace topology. Construct a topological model for the the quotient space Y of $[0, 1]$ where by the endpoints of the interval are identified.

25. According to the program above, the study of $[0, 1]$ should be equivalent to the study of the algebra of continuous, complex valued functions on $[0, 1]$. In fact, the quotient space corresponds to a subalgebra of $C([0, 1])$. Which subalgebra is it? Can you extrapolate a general statement from this example describing the subalgebra associated to a quotient space?

26. Let Y denote the space of binary sequences $y = (y_1, y_2, y_3, ...)$, where $y_i \in \{0, 1\}$ for each $i \in \mathbb{N}$. Show that $d(y, z) = \sum 2^{-k} |y_k - z_k|$ defines a metric on Y.

27. Define a relation on Y as follows. For each $y, z \in Y$, $y \sim z$ if and only if there exists $k \in \mathbb{N}$ such that $y_j = z_j$ for each $j \geq k$. Check that $\sim$ defines an equivalence relation on Y. Let $y \in Y$, and define the *orbit* of y under the relation $\sim$ to be $\{z \in Y | y \sim z\}$. Show that for $y \in Y$, the orbit of y is a dense set in Y with respect to d.

28. Describe the topological quotient space X of Y for which all the elements of an orbit are identified for each orbit. Is it Hausdorff? Following the prescription you developed above, determine the subalgebra of $C(Y)$ associated to X up to isomorphism. Does it accurately describe the space X?

The principles studied in this project show that the study of compact Hausdorff spaces is equivalent to the study of commutative unital C^*-algebras. These ideas have been generalized in modern mathematics in two important ways. In algebraic geometry, the theory of schemes takes *any* commutative ring with identity and makes it the ring of functions on a suitable space. This applies in particular to the ring of integers and its relatives and the resulting perspective has had a profound impact on number theory.

Alternatively, the base field remains $\mathbb{C}$, but the algebras are allowed to be non-commutative. The correspondence gives rise to so-called "non-commutative spaces," of which the last problem is an example. Fields Medalist Alain Connes coined this term for a space which is best described by the study of an associated non-commutative C^*-algebra, and non-commutative geometry is a developing field aimed at understanding this correspondence.