Group Theoretical Techniques in Differential Equations

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The goal of this project is to introduce Lie groups and their application for solving ordinary differential equations (ODEs). Amazingly enough, most of the 'recipe-based' techniques for solving differential equations that are typically taught are merely special cases of this much more general scheme that simply exploits the symmetries of the differential equation. It was invented by Sophus Lie in the 19th century and involves a beautiful application of algebra in analysis.

The project ends with a concrete example, the *Riccati* class of equations,

$$\frac{dy}{dx} = q_0 + q_1 y + q_2 y^2 \tag{1}$$

where q_0, q_1, q_2 are functions of x only. The Riccati equations have many important applications, including classical physics, variational calculus, nonlinear physics, thermodynamics, quantum theory, and even cosmology.

Lie Groups for first order ODEs

Consider the first order ODE

$$\frac{dy}{dx} = \omega(x, y) \tag{2}$$

1. A *symmetry* of Eq. (2) is a transformation $(x, y) \mapsto (\hat{x}, \hat{y})$ such that

   $$\frac{d\hat{y}}{d\hat{x}} = \omega(\hat{x}, \hat{y}) \tag{3}$$

   whenever Eq. (2) is satisfied. Namely, a transformation is a symmetry if it preserves the form of the original equation.

   Use the chain rule to prove that the transformation $(x, y) \mapsto (\hat{x}, \hat{y})$ is a symmetry of Eq. (2) if and only if the *symmetry condition* is satisfied

   $$\frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y} = \omega(\hat{x}, \hat{y}) \tag{4}$$

   Here the subscript denotes partial derivative.

2. A *one-parameter (local) Lie group* of Eq. (2) is a map

$$\Gamma_\varepsilon : (x, y) \mapsto (\hat{x}(x, y; \varepsilon), \hat{y}(x, y; \varepsilon)) \tag{5}$$

that satisfies the following conditions:

   - Γ_ε is a symmetry of Eq. (2) for every ε in some neighborhood of zero.
   - Γ_0 is the trivial symmetry, namely $(\hat{x}, \hat{y}) = (x, y)$ when $\varepsilon = 0$.
   - $\Gamma_\delta \Gamma_\varepsilon = \Gamma_{\delta+\varepsilon}$ for every δ, ε sufficiently close to zero.
   - $(\hat{x}, \hat{y})$ may be represented as a Taylor series in ε (about $\varepsilon = 0$).

   Prove that translations in the y direction $\Gamma_\varepsilon : (x, y) \mapsto (x, y + \varepsilon)$ satisfy the last three conditions in the definition of a one-parameter Lie group of Eq. (2). (The first condition depends on the equation studied.)

3. Prove that if the symmetries of Eq. (2) include the Lie group of translations in the y direction, then

$$\omega(x, y) = \omega(x, y + \varepsilon) \tag{6}$$

   for all ε in some neighborhood of zero.

4. Use the previous problem to show that in this case, the ODE in Eq. (2) can be solved immediately, and its general solution is

$$y = \int \omega(x) dx + c \tag{7}$$

   It is easy to see that in fact the one-parameter Lie group of y-translations acts on the set of solution curves by changing the constant of integration.

   This demonstrates that in fact, if one finds a 'good' coordinate system in which the symmetries are equivalent to translations, then the ODE can be easily solved.

   But what is a 'good' coordinate system? In the next part we will see that the Lie group contains the answer to this question as well.

Canonical Coordinates

Due to the properties of a one-parameter Lie group, one may write

$$\begin{aligned} \hat{x} &= x + \varepsilon\zeta(x,y) + O(\varepsilon^2) \\ \hat{y} &= y + \varepsilon\eta(x,y) + O(\varepsilon^2) \end{aligned} \tag{8}$$

It is useful to study the action of a one-parameter Lie group of symmetries on points in the $x-y$ plane. The *orbit* of the group through (x,y) is the set of points to which (x,y) can be mapped by varying ε.

It is convenient to think of the Lie group as describing a steady flow of particles on the plane, where the parameter ε represents "time", and then the tangent vector at (x,y)

$$(\zeta(x,y),\eta(x,y)) = \left(\left.\frac{d\hat{x}}{d\varepsilon}\right|_{\varepsilon=0}, \left.\frac{d\hat{y}}{d\varepsilon}\right|_{\varepsilon=0} \right) \tag{9}$$

represents the velocity of a particle at that point. In this analogy the orbit is the path of the particle.

A point that is mapped to itself by the Lie symmetries is called an *invariant point*. Similarly, a curve in the $x-y$ plane that is mapped to itself by the Lie symmetries is called an *invariant curve*.

The partial differential operator

$$\mathbb{X} = \zeta(x,y)\partial_x + \eta(x,y)\partial_y \tag{10}$$

is called the *infinitesimal generator* of the Lie group.

5. Show that all the orbits of the y-translation symmetries in Eq. (6) have the same tangent vector at every point:

$$(\zeta(x,y),\eta(x,y)) = (0,1) \tag{11}$$

6. Considering the problems in the first section, our goal is to introduce coordinates

$$(r,s) = (r(x,y), s(x,y)) \tag{12}$$

such that

$$(\hat{r},\hat{s}) \equiv (r(\hat{x},\hat{y}), s(\hat{x},\hat{y})) = (r, s+\varepsilon) \tag{13}$$

If this is possible, then in the new coordinates the tangent vector at the point (r, s) is $(0, 1)$, namely

$$\left(\left.\frac{d\hat{r}}{d\varepsilon}\right|_{\varepsilon=0}, \left.\frac{d\hat{s}}{d\varepsilon}\right|_{\varepsilon=0} \right) = (0, 1) \tag{14}$$

Use the chain rule to show that in this case the functions $r(x, y)$, $s(x, y)$ satisfy the linear partial differential equations

$$\begin{aligned} \mathbb{X}r = \zeta(x, y)r_x + \eta(x, y)r_y &= 0 \\ \mathbb{X}s = \zeta(x, y)s_x + \eta(x, y)s_y &= 1 \end{aligned} \tag{15}$$

Any pair of functions $r(x, y)$, $s(x, y)$ satisfying Eqs. (15) is called a pair of *canonical coordinates*. Note that canonical coordinates cannot be defined at an invariant point because the determining equation for s has no solution in this case.

Once Eqs. (15) are solved, one can use the new variables (r, s) to easily integrate the original ODE. Of course, in order to use this method one must first determine the canonical coordinates by solving the PDE (15) using the method of characteristics. Typically, this is much easier than solving the original ODE (2). An explicit example is outlined in the next section.

7. (A tough one...) Show that a curve $y(x)$ is an invariant curve if and only if it satisfies the *characteristic equation*

$$\eta(x, y) - \omega(x, y)\zeta(x, y) = 0 \tag{16}$$

Hint: What should be the direction of the flow for the curve $y(x)$ to be invariant?

Note that this is an *algebraic equation* for $y(x)$. This is a remarkable result that shows that one can sometimes obtain particular solutions of differential equations without doing a single integral!

Example

8. Consider the Riccati equation

$$\frac{dy}{dx} = xy^2 - \frac{2y}{x} - \frac{1}{x^3} \tag{17}$$

corresponding to the choice $q_0 = -1/x^3$, $q_1 = -2/x$ and $q_2 = x$ in Eq. (1). Show that the transformation

$$(\hat{x}, \hat{y}) = (e^{\varepsilon} x, e^{-2\varepsilon} y) \tag{18}$$

is a one parameter Lie group of scaling symmetries for this Riccati equation. Why are there exponentials in the definition of this Lie group?

9. Use the one parameter Lie group from the previous problem to show that it has two invariant solutions

$$y = \pm x^{-2} \tag{19}$$

10. Use the method of characteristics to solve the defining Eqs. (15) for canonical coordinates. Show that

$$(r, s) = \big(x^2 y, \ln|x|\big) \tag{20}$$

is a solution for $x \neq 0$.

11. Show that in these canonical coordinates the Riccati Eq. (17) reduces to

$$\frac{ds}{dr} = \frac{1}{r^2 - 1} \tag{21}$$

12. Integrate the last equation to show that in the original (x, y) coordinates, its general solution is

$$y = \frac{c + x^2}{x^2(c - x^2)} \tag{22}$$

In what limits are the invariant solutions curve from problem 9 can be obtained?