PERIODIC CONTINUED FRACTIONS

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Abstract. *The goals of this project are to have the reader explore some of the basic properties of continued fractions and prove that $\alpha \in \mathbb{R}$ is a quadratic irrational iff α is equal to a periodic continued fraction.*

1. Finite Continued Fractions

Fix $s = (a_0, (a_1, \ldots, a_n)) \in \mathbb{Z} \times \mathbb{N}^n$. The **finite (simple) continued fraction** of s is defined as

$$[s] = [a_0; a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}},$$

and if $n \geq k \in \mathbb{N}_0$, the kth **convergent** c_k of s is taken as $c_k = [a_0; a_1, \ldots, a_k]$. Prove that $k \in \{0, \ldots, n\} \Rightarrow$

$$c_k = \frac{p_k}{q_k} \tag{1.1}$$

where $p_0 = a_0$, $p_1 = a_0 a_1 + 1$, $q_0 = 1$, $q_1 = a_1$, and

$$p_m = a_m p_{m-1} + p_{m-2}, \tag{1.2}$$

$$q_m = a_m q_{m-1} + q_{m-2} \tag{1.3}$$

for all $m \in \{2, \ldots, n\}$. Next, use 1.1 and the definitions of p_m, q_m to show that

$$c_{k-1} - c_k = \frac{(-1)^k}{q_k q_{k-1}} \tag{1.4}$$

for all $k \in \{1, \ldots, n\}$ (Hint: Subtract q_{m-1} times 1.2 from p_{m-1} times 1.3.). Now employ 1.4 to conclude

$$c_0 \leq c_2 \leq c_4 \leq \ldots \leq [s] \leq \ldots \leq c_5 \leq c_3 \leq c_1. \tag{1.5}$$

Also, use 1.4 again to demonstrate that $(p_k, q_k) = 1$ for all $k \in \{0, \ldots, n\}$.

2. Infinite Continued Fractions

Fix $t = (a_0, (a_1, a_2, \ldots)) \in \mathbb{Z} \times \mathbb{N}^\infty$ and extend the definitions of c_k, p_k, q_k to all $k \in \mathbb{N}_0$. Prove that the limit (called the **infinite (simple) continued fraction** of t)

$$\lim_{k \to \infty} c_k = [t] = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

exists as follows: First, note that 1.5 implies the limits $\lim_{k\to\infty} c_{2k}, \lim_{k\to\infty} c_{2k+1}$ exist by the monotone convergence theorem. Next, argue it's enough to show these limits are equal, which amounts to proving $q_{2k+1}q_{2k} \to \infty$ as $k \to \infty$ by 1.4. Lastly, complete the proof by establishing the estimate $q_k \geq k$ for all $k \in \mathbb{N}_0$.

Now show that $[t] \notin \mathbb{Q}$. Suppose $[t] = n_0/d_0$ for some $n_0, d_0 \in \mathbb{Z}$ with $d_0 > 0$. Obtain a contradiction by showing that for each $k \in \mathbb{N}_0$ there are $n_k, d_k \in \mathbb{Z}$ such that $r_k = n_k/d_k$ and $d_0 > d_1 > \ldots > d_{k-1} > d_k > 0$ where r_k is the kth **remainder** of t defined as $r_k = [a_k; a_{k+1}, a_{k+2}, \ldots]$.

Next, fix $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Let $a_0' = \lfloor \alpha \rfloor \in \mathbb{Z}$ be the greatest integer less than or equal to α. Then $\alpha - a_0' > 0$ since $\alpha \notin \mathbb{Q}$, so one may define $a_1' = \lfloor (\alpha - a_0')^{-1} \rfloor \in \mathbb{N}$. Likewise, $(\alpha - a_0')^{-1} - a_1' > 0$, so one may define $a_2' = \lfloor ((\alpha - a_0')^{-1} - a_1')^{-1} \rfloor \in \mathbb{N}$. Iterating this process gives a sequence $t' = (a_0', (a_1', a_2', \ldots)) \in \mathbb{Z} \times \mathbb{N}^\infty$. Deduce that $\alpha = [t']$ (Hint: Let c_k' denote the kth convergent of t' and observe that by construction $\alpha > c_{2k}'$ and $\alpha < c_{2k+1}'$ for all $k \in \mathbb{N}_0$, so the result follows by the squeeze theorem). Finally, prove $[t] = [t'] \Rightarrow t = t'$. Thus all real irrational numbers are equal to the infinite (simple) continued fraction of a uniquely determined sequence in $\mathbb{Z} \times \mathbb{N}^\infty$.

3. Characterization of Quadratic Irrationals

Fix $t = (a_0, (a_1, a_2, \ldots)) \in \mathbb{Z} \times \mathbb{N}^\infty$ as above and denote its convergents and remainders as before by $c_k = p_k/q_k$ and r_k, respectively. Use the relation $r_k = (r_{k-1} - a_{k-1})^{-1}$ for all $k \in \mathbb{N}$ to prove the following lemma.

Lemma. $2 \leq k \in \mathbb{N} \Rightarrow$
$$[t] = \frac{r_k p_{k-1} + p_{k-2}}{r_k q_{k-1} + q_{k-2}}.$$

We say that $[t]$ is a **periodic continued fraction** iff there are $m, n \in \mathbb{N}$ such that $m \leq k \in \mathbb{N} \Rightarrow a_k = a_{k+n}$; this is a well-defined notion by uniqueness. Also, we say that α is a **quadratic irrational** iff $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and α is the root of a quadratic polynomial with integer coefficients. Somewhat surprisingly, these concepts are related in the same way that repeating decimal expansions are related to rationality; in particular, we have the following theorem. Fill in the details of the proof.

Theorem. *$\alpha \in \mathbb{R}$ is a quadratic irrational iff α is equal to a periodic continued fraction.*

Proof. ($\Rightarrow$) Suppose $\alpha \in \mathbb{R}$ is a quadratic irrational. Then we've seen that α is equal to the infinite continued fraction of a uniquely determined sequence in $\mathbb{Z} \times \mathbb{N}^\infty$, so wlog $\alpha = [t]$. Also, there are $a, b, c \in \mathbb{Z}$ such that $a\alpha^2 + b\alpha + c = 0$. An application of the above lemma and a tedious calculation shows that $2 \leq k \in \mathbb{N} \Rightarrow A_k r_k^2 + B_k r_k + C_k = 0$ where $A_k, B_k, C_k \in \mathbb{Z}$ with

$$A_k = C_{k+1} = \alpha p_{k-1}^2 + b p_{k-1} q_{k-1} + c q_{k-1}^2, \tag{3.1}$$
$$B_k^2 - 4A_k C_k = b^2 - 4ac. \tag{3.2}$$

In addition, utilizing the lemma again along with 1.2, 1.3, and 1.4, gives that $3 \leq k \in \mathbb{N} \Rightarrow$
$$\left| \alpha - \frac{p_{k-1}}{q_{k-1}} \right| = \frac{r_{k-1} - a_{k-1}}{(r_{k-1} q_{k-2} + q_{k-3}) q_{k-1}} < \frac{1}{q_{k-1}^2}.$$

Thus by taking $\delta_{k-1} = p_{k-1}q_{k-1} - \alpha q_{k-1}^2$ for $3 \le k \in \mathbb{N}$ we have $p_{k-1} = \alpha q_{k-1} + \delta_{k-1}/q_{k-1}$ with $|\delta_{k-1}| < 1$, so 3.1 implies

$$|A_k| = |C_{k+1}| = \left| 2a\alpha\delta_{k-1} + a\frac{\delta_{k-1}^2}{q_{k-1}^2} + b\delta_{k-1} \right| < 2|a\alpha| + |a| + |b| < \infty. \tag{3.3}$$

Hence $|\{(A_k, B_k, C_k) | 2 \le k \in \mathbb{N}\}| < \infty$ since 3.3 shows A_k, C_k are bounded sequences of integers and, consequently, 3.2 shows B_k is a bounded sequence of integers. Therefore $|\{r_k | k \in \mathbb{N}\}| < \infty$, so there are $m, n \in \mathbb{N}$ such that $r_m = r_{m+n}$, giving $a_k = a_{k+n}$ whenever $m \le k \in \mathbb{N}$.

($\Leftarrow$) Conversely, suppose $\alpha = [t]$ is a periodic continued fraction, so $\exists m, n \in \mathbb{N}$ such that $m \le k \in \mathbb{N} \Rightarrow a_k = a_{k+n}$. It follows that $m \le k \in \mathbb{N} \Rightarrow r_k = r_{k+n}$, so by the lemma we have

$$\frac{r_m p_{m-1} + p_{m-2}}{r_m q_{m-1} + q_{m-2}} = \frac{r_m p_{m+n-1} + p_{m+n-2}}{r_m q_{m+n-1} + q_{m+n-2}},$$

whence r_m is a quadratic irrational since we know infinite continued fractions are irrational from the previous section. On the other hand, the reciprocal of a quadratic irrational is a quadratic irrational and the sum of an integer plus a quadratic irrational is a quadratic irrational, so r_{m-1} is a quadratic irrational. Inductively, $[t] = r_0$ is a quadratic irrational. $\square$

By the theorem we know that

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

is a quadratic irrational; in this case, it's easy to see that $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Recall that $F_n = [\phi^n - (1-\phi)^n]/\sqrt{5}$ for all $n \in \mathbb{N}_0$ where F_n is the nth Fibonacci number defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$. Less trivially, the theorem also shows that the continued fraction constant C given by

$$C = 0 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cdots}}}$$

is either transcendental or algebraic of degree greater than or equal to 3. In general, however, infinite continued fractions are rather mysterious animals and there are no analogous characterizations for transcendentals or algebraic numbers of degree higher than 2, although there are beautiful non-simple continued fraction expansions for transcendentals as well as the approximations $\gamma \approx [0, 1, 1, 2, 1, 2, 1, 4]$, $e \approx [2, 1, 2, 1, 1, 4, 1, 1]$, and $\pi \approx [3, 7, 15, 1, 292, 1, 1, 1]$ where γ is Euler's constant. Moreover, as of the early 21st century not one (simple) continued fraction expansion has yet been completely determined for any numbers other than rationals and quadratic irrationals. So little is known, in fact, that deciding whether or not the elements of the infinite continued fraction of $\sqrt[3]{2}$ are bounded would be a major discovery.

References

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[2] A.Ya. Kinchin, *Continued Fractions*, Dover, Mineola, 1997.