INTEGRATION WORKSHOP PROJECT: TOPOLOGICAL GROUPS AND DUALITY

Abstract. Topological groups represent (in a somewhat surprisingly rigorous sense) objects in the intersection of the theory of groups with the theory of topological spaces. Given the vast literature on both of these topics, it is perhaps not surprising that we can obtain significantly stronger results about objects to which both of these studies apply. Thus it is common in algebra and number theory to equip groups of interest with topologies, and similarly common in topology and geometry to equip topological spaces with group structures. The goal of this project is to introduce topological groups and their duals, and explore what happens to nice topological properties upon dualizing.

Definition 1. A *topological group* is a triple $(G, \cdot, T)$ where G is a set, $\cdot$ is a group operation on G, and T is a topology on G such that the operations of group multiplication and taking inverses are continuous maps, i.e. such that the following two maps are continuous with respect to T:

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto gh, \end{aligned}$$

(where $G \times G$ is given the product topology), and

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1}. \end{aligned}$$

Abusing notation, we will usually refer to such a topological group just as G. It's worth noting here that, thought we don't require it here, the definition of a topological group frequently mandates that the equipped topology be Hausdorff. We'll see in Problem 7 that this is a fairly mild (and useful!) hypothesis.

There are several quick examples of topological groups, stemming from many of the objects we are accustomed to thinking about as both groups and topological spaces:

Example 2. The following are topological groups:

- Any group with the discrete topology.
- Any group with the indiscrete topology.
- The groups $\mathbb{R}^n$, $\mathbb{R}^\times$, $\mathbb{R}_+^\times$, $\mathbb{C}^n$, and $\mathbb{C}^*$, each equipped with the Euclidean topology.
- The complex unit circle S^1 with the induced Euclidean topology.
- The matrix groups $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ (though these require some checking!)

Problem 3. *Prove, or at least convince yourself, that all of the above are indeed topological groups.*

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Problem 4. *We've just seen above that groups (for example, $\mathbb{R}$) can often be equipped with various topologies in order to turn it in to a topological group. Let $G = \{e, g\}$ be the group with 2 elements. How many topological group structures (up to homeomorphism) can be placed on G? (Compare with Problem 1.1 of the topology problem set).*

One particularly nice aspect of working with topological groups is that we can take advantage of the continuity of multiplication to get global results from information at a single point. As a first stab at exploiting this fact, prove:

Problem 5. *The topology on G is completely determined by a basis for the neighborhoods of the identity element.*

And as our first application of this technology:

Proposition 6. *A topological group G is Hausdorff if and only if the one-point set $\{e\}$ consisting of the identity element is closed (compare with the topology exercise 1.3 to see that this statement is false if G is merely a topological space).*

Problem 7. *Prove it! One direction follows easily from the topology exercise 1.3. For the other, start by proving the following lemma, and use it to "separate" the identity.*

Lemma 8. *If $U \subset G$ is a neighborhood of the identity, then there exists a sub-neighborhood $V \subset U$ such that $VV \subseteq U$.*

Definition 9. The *Pontrjagin dual* $\widehat{G}$ of an abelian topological group G is defined to be set of all continuous homomorphisms (called *characters*) from G into the complex unit circle (denoted S^1), i.e.,

$$\widehat{G} := \mathrm{Hom}(G, S^1).$$

The group operation on $\widehat{G}$ is pointwise multiplication, and the identity element is the trivial character ι defined by $\iota(g) = 1$ for all $g \in G$.

Problem 10. *For each of the following G, describe $\widehat{G}$:*

- $G = \mathbb{Z}$.
- $G = S^1$.
- $G = \mathbb{Z}/p^k\mathbb{Z}$.
- *An arbitrary finite abelian group.*
- $G = D_3$, *the dihedral group of 6 elements (note this is non-abelian, so we'll have to attempt a suitable extension of our definition of the dual group)*

Of course, if we want to discuss topological properties of the dual of a group, we had better first establish a topology on it.

Definition 11. We define the *compact-open* topology on the dual of an abelian topological group. Let K be a compact subset of G and let V be a neighborhood of the identity in S^1. Define

$$W(K, V) = \{\chi \in \widehat{G} : \chi(K) \subset V\},$$

and take these sets to be a basis for the neighborhoods of $\iota \in \widehat{G}$. By problem 5, this determines a topology on all of $\widehat{G}$.

Problem 12. *Describe this topology more explicitly in the case where G is equipped with the discrete topology.*

Your final problem is to fill in the outline of the proof of our last theorem.

Theorem 13. *Let G be an abelian topological group. Then*

$$G \text{ discrete} \quad \Rightarrow \quad \widehat{G} \text{ compact}$$

and

$$G \text{ compact} \quad \Rightarrow \quad \widehat{G} \text{ discrete}$$

Proof outline.

- Verify this for the examples given in problem 10.
- For the first implication, consider $\widehat{G}$ as a subset of *all* maps from G to S^1, and use the fact that a closed subset of a compact space is itself compact.
- For the second implication, show that the set consisting of only the trivial character is open.

□