INTEGRATION WORKSHOP 2006
GENERATING FUNCTIONS

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1. Solving Linear Recurrences using Generating Functions

A *linear recurrence* is any recursive definition for a sequence $\{a_n\}$ of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

It is said to have *order* k, and evidently one must specify the first k values $a_0, a_1, \ldots,$ a_{k-1} to obtain a unique sequence $\{a_n\}$ satisfying the recurrence (note that here we'll use the convention that the first term in a sequence is a_0). The following discussion is valid over any field for which the *characteristic polynomial*

$$1 - c_1 x - c_2 x^2 - \cdots - c_k x^k$$

factors into linear terms; but for simplicity, we will assume that all of the sequences are defined over the complex numbers $\mathbb{C}$.

For a sequence $\{a_n\}$ that satisfies a linear recurrence, show that the Taylor series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges in a neighborhood of $x = 0$. The function $f(x)$ is called a *generating function* for the sequence $\{a_n\}$. Show that in this neighborhood, $f(x) \cdot p(x) = q(x)$, where $p(x)$ is the characteristic polynomial of the recurrence and $q(x)$ is a polynomial of degree $< k$.

Now suppose that $p(x)$ has roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ and factors as $(x - \lambda_i)^{d_i}$ with $\lambda_i \in \mathbb{C}$. Show that there exist polynomials $r_i(x)$ with $\deg(r_i) < d_i$ so that

$$\frac{q(x)}{p(x)} = \sum_{i=1}^{k} \frac{r_i(x)}{(x - \lambda_i)^{d_i}}.$$

One approach would be to let V be the vector space of rational functions of the form $\frac{r(x)}{p(x)}$, where $\deg(r) < \deg(p)$, and then show that the functions

$$\frac{1}{(x - \lambda_i)^j}, \quad 1 \le j \le d_i$$

form a basis of V.

As an aside, the upshot of what we just proved was that the method of partial fractions "works": one can always decompose a rational function into a sum of a polynomial and a (unique) set of partial fractions over $\mathbb{C}$. In fact, if the rational function has coefficients in $\mathbb{R}$, one could then recombine the partial fractions corresponding to complex conjugate roots, and recover the usual partial fraction decomposition from integral calculus. An important corollary is the fact that every rational function has an antiderivative that can be expressed in terms of elementary functions.

Finally, use the Taylor series for

$$\frac{1}{(x-\lambda_i)^j}$$

to obtain a closed-form expression for the sequence a_n. *Hint:* you can find the Taylor series for the above expression by successively differentiating term-by-term the well-known Taylor series for $\frac{1}{(x-\lambda_i)}$.

Apply these ideas to find closed-form expressions for the following sequences:

- (The Fibonacci numbers) $F_0 = 0$; $F_1 = 1$; $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
- $a_0 = 1$; $a_1 = 3$; $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$.

2. Some Other Interesting Generating Functions (Optional)

Catalan Numbers. The Catalan numbers c_n arise in many combinatorial problems. One of the easiest ways to describe them is as the number of ways to arrange n pairs of parentheses. For example, $c_3 = 5$:

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By convention, $c_0 = 1$. Show that the Catalan numbers satisfy the recurrence

$$c_{n+1} = \sum_{i=0}^{n} c_i c_{n-i}.$$

Show that their generating function

$$C(x) = \sum_{n=0}^{\infty} c_n x^n$$

converges in a neighborhood of $x = 0$, and use the recurrence to find a formula for $C(x)$ valid in this neighborhood. *Hint:* careful consideration of $C(x)^2$ should lead to a quadratic equation in $C(x)$.

Use the binomial Taylor series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

to find a closed-form expression for c_n.

Derangements. A permutation π of the set $\{1, 2, \ldots, n\}$ such that $\pi(i) \neq i$ for all i is called a *derangement*. The number of derangements of a set of size n is denoted d_n, where $d_0 = 1$ by convention. Show that the number of derangements d_n satisfies the recurrence

$$d_{n+1} = n(d_n + d_{n-1}).$$

Hint: if π is a derangement of $\{1, 2, \ldots, n+1\}$, then $\pi(n+1) = i$ for some $i \in \{1, 2, \ldots, n\}$. Now consider separately the cases that $\pi(i) = n+1$ and $\pi(i) \neq n+1$.

There are too many derangements for the generating function of the sequence $\{d_n\}$ to converge in any neighborhood of $x = 0$; in fact, we will see below that d_n is of the same order of magnitude as $n!$. So instead we'll consider the *exponential generating function*

$$D(x) = \sum_{n=0}^{\infty} \frac{d_n}{n!} x^n.$$

Show that $D(x)$ converges in a neighborhood of $x = 0$, and use the recurrence for d_n to show that, in that neighborhood, $D(x)$ satisfies the differential equation

$$(1 - x)D'(x) = xD(x).$$

Solve this differential equation for $D(x)$, and use it to prove that

$$d_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

The probability p_n that a random permutation $\pi \in S_n$ is fixed-point-free is evidently $\frac{d_n}{n!}$. Find $\lim_{n\to\infty} p_n$.