Integration Workshop Project

1 Group Representations

A representation of a group G gives us a way of visualizing G as a group of matrices. To be precise, a representation is a homomorphism from G into a group of invertible matrices. We would like to study some basic ideas of representation theory and connect it to basic ideas from linear algebra.

Definition 1 *A representation of G over F is a homomorphism ρ from G to $GL(n, F)$, for some n. The degree of ρ is the integer n.*

Definition 2 *Let $\rho : G \to GL(m, F)$ and $\sigma : G \to GL(n, F)$ be representations of G. We say that ρ is equivalent to σ if $n = m$ and there exists and invertible $n \times n$ matrix T such that for all $g \in G$, $g\sigma = T^{-1}(g\rho)T$.*

Problem 3 *Show that equivalent representations form an equivalence class of representations. (i.e. show that it is reflexive, symmetric, and transitive).*

Problem 4 *Consider the representation of the dihedral group (symmetries on a square) $G = D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ where a is a rotation and b is a reflection with*

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *(consider a square with vertices $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$*

(i) What is another possible representation?

(ii) Is the following a representation?

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$?

Definition 5 *A representation $\rho : G \to GL(n, F)$ is said to be faithful if $\ker \rho = \{1\}$.*

Problem 6 *Prove the following: A representation ρ of a finite group G is faithful if and only if $\operatorname{Im} \rho$ is isomorphic to G. What are the dimensions of the faithful representations of D_8 above? Can a group have a faithful 1-dimensional representation? If not explain why. If possible give an example.*

Problem 7 *Let* $G = D_{12} = \langle a, b : a^6 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$, *the symmetries of a regular hexagon. Define the matrices* A, B, C, D *over* $\mathbb{C}$ *by*

$$A = \begin{pmatrix} e^{i\pi/3} & 0 \\ 0 & e^{-i\pi/3} \end{pmatrix}, \; B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Prove that each of the function $\rho_k : G \to GL(2, \mathbb{C})$ $(k = 1, 2, 3, 4)$, *given by*

$\rho_1 : a^r b^s \to A^r B^s$,

$\rho_2 : a^r b^s \to A^{3r} (-B)^s$

$\rho_3 : a^r b^s \to (-A)^r B^s$,

$\rho_4 : a^r b^s \to C^r D^s \qquad (0 \leq r \leq 5, 0 \leq s \leq 1)$,

is a representation of G. *Which of these representations are faithful? Which are equivalent?*

Problem 8 *Give an example of a faithful representation of* D_8 *of degree 3.*

Problem 9 *Suppose that* ρ *is a representation of* G *of degree* 1. *Prove that* $G/\ker\rho$ *is abelian.*

2 FG-modules

We now introduce the concept of an FG-module and show that there is a close connection between FG-modules and representations of G over F.

Let G be a group and let F be $\mathbb{R}$ or $\mathbb{C}$.

Suppose that $\rho : G \to GL(n, F)$ is a representation of G. Write $V = F^n$, the vector space of all row vectors $(\lambda_1, ..., \lambda_n)$ with $\lambda_i \in F$. For all $v \in V$ and $g \in G$ the matrix product

$$v(g\rho),$$

of the row vector v with the $n \times n$ matrix $g\rho$ is a row vector in V.

Some basic properties are

1. $v(1\rho) = v$,
2. $(\lambda v)(g\rho) = \lambda(v(g\rho))$, and
3. $(u + v)(g\rho) = u(g\rho) + v(g\rho)$ for all $u, v \in V$, $\lambda \in F$ and $g \in G$.

Problem 10 *Let* $G = D_8$ *with the representation as in problem4 above. If* $v = (\lambda_1, \lambda_2) \in F^2$ *then compute the following:* $v(a\rho)$, $v(b\rho)$, *and* $v(a^3\rho)$.

Motivated by these above observations on the product $v(g\rho)$ we now define an FG-module.

Definition 11 *Let* V *be a vector space over* F *and let* G *be a group. Then* V *is an* FG*-module if a multiplication* vg $(v \in V, g \in G)$ *is defined, satisfying the following conditions for all* $u, v \in V$, $\lambda \in F$, *and* $g, h \in G$

1. $vg \in V$;
2. $v(gh) = (vg)h$;
3. $v1 = v$;
4. $(\lambda v)g = \lambda(vg)$;
5. $(u + v)g = ug + vg$

We use the letters F and G in the name "FG-module" to indicate that V is a vector space over F and that G is the group from which we are taking the elements g to form the products vg.

Definition 12 *Let* V *be an* FG*-module, and let* $\mathcal{B}$ *be a basis of* V*. For each* $g \in G$, *let*

$$[g]_{\mathcal{B}}$$

denote the matrix of the endomorphism $v \to vg$ *of* V *relative to the basis* $\mathcal{B}$.

The connection between FG-modules and representations of G over F is revealed in the following theorem.

Theorem 13 *(1) If* $\rho : G \to GL(n, F)$ *is a representation of* G *over* F *and* $V = F^n$, *then* V *becomes an* FG*-module if we define the multiplication* vg *by*

$$vg = v(g\rho) \qquad (v \in V, g \in G).$$

Moreover there is a basis $\mathcal{B}$ *of* V *such that*

$$g\rho = [g]_{\mathcal{B}} \text{ for all } g \in G.$$

(2) Assume that V *is an* FG*-module and let* $\mathcal{B}$ *be a basis of* V*. Then the function* $g \to [g]_{\mathcal{B}}$ $(g \in G)$ *is a representation of* G *over* F.

As you might expect, there are restrictions on how we may define the vectors $v_i g$. So your job is to prove the next result which is often used to show that our chosen multiplication turns V into an FG-module.

Proposition 14 *Assume that* $v_1, ..., v_n$ *is a basis of a vector space* V *over* F*. Suppose that we have a multiplication* vg *for all* v *in* V *and* g *in* G *which satisfies the following conditions for all* i *with* $1 \le i \le n$, *for all* $g, h \in G$ *and for all* $\lambda_1, ..., \lambda_n \in F$:

1. $v_i g \in V$
2. $v_i(gh) = (v_i g)h$

3. $v_i 1 = v_i$

4. $(\lambda_1 v_1 + ... + \lambda_n v_n) g = \lambda_1 (v_1 g) + ... + \lambda_n (v_n g)$.

Then V is an FG-module.

Problem 15 *Prove it!*

Definition 16 *Let G be a subgroup of S_n. The FG-module V with basis $v_1, ..., v_n$ such that*

$$v_i g = v_{ig} \text{ for all } i \text{ and all } g \in G$$

is called the permutation module for G over F. We call $v_1, ..., v_n$ the natural basis of V.

Example 17 *Let $G = S_4$ and let $\mathcal{B}$ denote the basis v_1, v_2, v_3, v_4 of V. If $g = (12)$, then*

$$v_1 g = v_2,\ v_2 g = v_1,\ v_3 g = v_3,\ v_4 g = v_4.$$

And if $h = (134)$, then

$$v_1 h = v_3,\ v_2 h = v_2,\ v_3 h = v_4, \text{ and } v_4 h = v_1$$

We have thus

$$[g]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } [h]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Problem 18 *Suppose that $G = S_3$ and that $V = \operatorname{span}(v_1, v_2, v_3)$ is the permutation module for G over $\mathbb{C}$. Let $\mathcal{B}_1$ be the basis v_1, v_2, v_3 of V and let $\mathcal{B}_2$ be the basis $v_1 + v_2 + v_3, v_1 - v_2, v_1 - v_3$. Calculate the 3×3 matrices $[g]_{\mathcal{B}_1}$ and $[g]_{\mathcal{B}_2}$ for all g in S_3. What do you notice about the matrices $[g]_{\mathcal{B}_2}$?*

Problem 19 *Let A be an $n \times n$ matrix and let B be a matrix obtained from A by permuting the rows. Show that there is an $n \times n$ permutation matrix P such that $B = PA$. Find a similar result for a matrix obtained from A by permuting the columns.*

3 FG-submodules and reducibility

Definition 20 *Let V be an FG-module. A subset W of V is said to be an FG-submodule of V if W is a subspace and $wg \in W$ for all $w \in W$ and all $g \in G$.*

Definition 21 *An FG-module V is said to be irreducible if it is non-zero and it has no FG-submodules apart from $\{0\}$ and V.*

Problem 22 *Let* $G = C_3$ *the cyclic group of order* $3 = \langle a : a^3 = 1 \rangle$, *and let* V *be the* 3 *dimensional* FG*-module given by* $v_1 1 = v_1$, $v_2 1 = v_2$, $v_3 1 = v_3$, $v_1 a = v_2$, *etc. Is* $\operatorname{span}(v_1 + v_2)$ *an* FG*-submodule? Is this* FG*-module reducible or irreducible? Prove your answer.*

Suppose that V is a reducible FG-module, so that there is an FG-submodule W with $0 < \dim W < \dim V$. Take a basis $\mathcal{B}_1$ of W and extend it to a basis $\mathcal{B}$ of V. Then for all g in G the matrix $[g]_{\mathcal{B}}$ has the form

$$\begin{pmatrix} X_g & 0 \\ Y_g & Z_g \end{pmatrix}$$

for some matrices X_g, Y_g, and Z_g, where X_g is $k \times k$ ($k = \dim W$). A representation is reducible if and only if it can be put in this form.

Problem 23 *Let* $G = C_2 = \langle a : a^2 = 1 \rangle$ *and let* $V = F^2$. *For* $(\alpha, \beta) \in V$, *define* $(\alpha, \beta) 1 = (\alpha, \beta)$ *and* $(\alpha, \beta) a = (\beta, \alpha)$. *Verify that* V *is an* FG*-module and find all the* FG*-submodules of* V.

Problem 24 *Let* ρ *and* σ *be equivalent representations of they group* G *over* F. *Prove that if* ρ *is reducible then* σ *is reducible.*

Problem 25 *Define the permutation* $a, b, c \in S_6$ *by*

$$a = (123), \; b = (4,56), \; \text{and } c = (23)(45),$$

and let $G = \langle a, b, c \rangle$, *the group generated by these three permutations.*

(a) Check that

$a^3 = b^3 = c^2 = 1$, $ab = ba$, $c^{-1}ac = a^{-1}$ *and* $c^{-1}bc = b^{-1}$. *Deduce that* G *has order* 18.

(b) Suppose that ε *and* η *are complex cube roots of unity. Prove that there is a representation* ρ *of* G *over* $\mathbb{C}$ *such that*

$$a\rho = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \; b\rho = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}, \; c\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(c) For which values of ε, η *is* ρ *faithful?*

(d) For which values of ε, η *is* ρ *irreducible?*

Problem 26 *Let* $G = C_{13}$. *Find a* $\mathbb{C}G$*-module which is neither reducible nor irreducible.*

The following lemma, Schur's lemma is a basic result concerning irreducible modules. It is a fundamental result to representation theory and there is an immediate application which helps to determine all the irreducible representations of finite abelian groups. Note that Schur's lemma deals with $\mathbb{C}G$-modules.

Lemma 27 *(Schur's) Let* V *and* W *be irreducible* $\mathbb{C}G$*-modules.*

(1) If $\theta : V \to W$ *is* $\mathbb{C}G$*-homomorphism, then either* θ *is a* $\mathbb{C}G$*-isomorphism, or* $v\theta = 0$ *for all* $v \in V$.

(2) If $\theta : V \to V$ *is a* $\mathbb{C}G$*-isomorphism, then* θ *is a scalar multiple of the identity endomorphism* 1_V.