INTEGRATION WORKSHOP 2003
PROJECT ON CALCULATING RATIONAL CANONICAL FORMS

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We know every square matrix is equivalent to a matrix in **Rational Canonical Form** (RCF). In this project, we consider ways of calculating the RCF of a matrix.

1. Low Dimensions

Explain how, in dimension 3 or less, knowing the minimal polynomial and the Generalized Cayley-Hamilton Theorem is enough to calculate the RCF.

2. Theory of Rational Canonical Forms

The theory of RCF as we have covered it in the lectures goes like this: given a vector space V and a linear transformation T, we regard V as a module over $K[t]$ via T, express V as the cokernel of a map

$$N : K[t]^n \to K[t]^m$$

for some $n \times m$ matrix N, then put the matrix N in Smith Normal Form, i.e.

(1) every entry off the main diagonal of N is 0;
(2) on the main diagonal of N there appear (in order) polynomials $f_1, ..., f_\ell$ such that f_k divides f_{k+1}, $1 \leq k \leq \ell - 1$ and $\ell = \min(m, n)$.

To get an algorithm, we need to make all the steps in this proof explicit. Suppose that $V = K^n$ and that T is given by the matrix A. Show that we can choose $N = A - tI$.

3. Matrices in Rational Canonical Form

Now we want to find a way of diagonalizing $A - tI$. As a warm-up exercise, consider the case where A is the companion matrix for the polynomial $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$,

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}.$$

Show that $p(t)$ is the characteristic and minimal polynomial of its companion matrix. Now diagonalize $A - tI$, where A is an arbitrary matrix, using the following operations:

(1) multiplication of one row/column of A by a non-zero scalar in K;
(2) replacement of the rth row/column of A by row/column r plus f times row/column s, where f is any polynomial over K and $r \neq s$;
(3) interchange of two rows/columns of A.

Once you have figured out how to handle the case where A is a companion matrix, explain how to handle the case where A is any matrix in Rational Canonical Form.

4. Diagonalization in General

Using the fact that every matrix can be put in RCF, explain how to put the matrix $A - tI$ in Smith Normal Form for a general matrix A.

5. Practical Algorithm

Find a practical algorithm to transform $A - xI$ into the matrix N. By practical, I mean an algorithm you could use to find N for a 10×10 matrix over $\mathbb{Q}$ by hand. Let P be the matrix of row operations and Q be the matrix of column operations performed above. What is the significance of the matrices P and Q?