INTEGRATION WORKSHOP PROJECT: THE WEIERSTRASS $\mathcal{P}$-FUNCTION

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1. Elliptic functions.

Let ω be a non-zero complex number. A function $f(z)$ is called *periodic* with period ω if

$$f(z) = f(z+\omega)$$

for all z. For example, $\exp(z)$ is periodic with period $2\pi i$, and both $\sin z$ and $\cos z$ are periodic with period 2π.

Let D be a domain such that for all $z \in D$ we have $z+\omega \in D$ and $z-\omega \in D$. Let D_1 be the image of D under the map $z \mapsto \exp(2\pi i z/\omega)$. Let $f(z)$ be meromorphic in D with period ω.

1. Show that there exists a unique function g in D_1 such that

$$f(z) = g(\exp(2\pi i z/\omega)).$$

Let now $f(z)$ be a non-constant meromorphic function defined in $D = \mathbb{C}$. Let M be the set of periods of $f(z)$, i.e. those complex numbers $\omega \in \mathbb{C}$ that satisfy $f(z+\omega) = f(z)$ for all z. It is clear that once $\omega \in M$, then all its integral multiples $n\omega$, $n \in \mathbb{Z}$ are also in M. Although zero is not really a period, we add it to M. If M has numbers other than zero in it, choose one, ω_1, with the smallest positive absolute value. Now we assume that there are periods in M other than the integral multiples of ω_1. Choose one of them, ω_2, with the smallest absolute value. You should verify that under our assumptions the ratio ω_1/ω_2 is not real. In such a case we say that $f(z)$ is *doubly periodic*, i.e. there exist two non-zero complex numbers ω_1 and ω_2 such that:

a) $f(z+\omega_1) = f(z)$ for all z

b) $f(z+\omega_2) = f(z)$ for all z and

c) ω_1/ω_2 is not real

Our additional assumptions on the absolute values of ω_1 and ω_2, however, amount to the following statement:

2. Show that all the periods of $f(z)$ are of the form $n_1\omega_1 + n_2\omega_2$ with n_1, $n_2 \in \mathbb{Z}$.

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In this case, when all the periods of a doubly periodic function are of the form $n_1\omega_1 + n_2\omega_2$ we say that (ω_1, ω_2) is a *basis* of M.

3. Show that if (η_1, η_2) is another basis of M, then there exist integer numbers a, b, c, and d such that

$$\eta_1 = a\omega_1 + b\omega_2$$

$$\eta_2 = c\omega_1 + d\omega_2$$

$$|ad - bc| = 1.$$

The doubly periodic functions are also called *elliptic functions*.

4. Show that if an elliptic function has no poles, it is bounded in $\mathbb{C}$ and thus by Liouville's theorem must be constant.

We can choose a number $a \in \mathbb{C}$ such that $f(z)$ has no zeroes or poles on the boundary ∂P of the parallelogram P with vertices at the points a, $a + \omega_1$, $a + \omega_2$, $a + \omega_1 + \omega_2$.

5. Show that the double periodicity of $f(z)$ implies that

$$\int_{\partial P} f(z)\, dz \;=\; 0.$$

As a consequence, the sum of residues of $f(z)$ is zero and therefore there doesn't exist an elliptic function with a single simple pole inside P.

6. Show that the function $f'(z)/f(z)$ is also an elliptic function and deduce that $f(z)$ has equally many poles as it has zeroes inside P.

7*. By analyzing the integral

$$\frac{1}{2\pi i}\int_{\partial P} \frac{zf'(z)}{f(z)}\, dz$$

show that on one hand its value is of the form $n_1\omega_1 + n_2\omega_2$, and on the other hand equals $a_1 + \cdots + a_k - b_1 - \cdots - b_k$, where $a_1, ..., a_k$ are the zeroes of $f(z)$ inside P and $b_1, ..., b_k$ are the poles. Conclude that the sum of zeroes minus the sum of poles (as complex numbers) is a period of $f(z)$.

2. Weierstrass $\mathcal{P}$-function.

Let, as before, $f(z)$ be a non-constant elliptic function with a basis of periods (ω_1, ω_2). Assume now that P contains the origin and that the only pole of $f(z)$ inside P is actually at the origin. Since it cannot be a simple pole, the next best thing is to require that the singular part of $f(z)$ is z^{-2}.

8. Show that $f(z) - f(-z)$ has no poles and thus is a constant function, which is moreover zero. This proves that $f(z)$ is an even function.

Next, we can assume, by adding a constant if necessary that the Laurant series for $f(z)$ at zero has no constant term:

$$f(z) = z^{-2} + a_1 z^2 + a_2 z^4 + \cdots .$$

With these conditions, it is easy to see that $f(z)$ is uniquely determined and it is traditionally denoted by $\mathcal{P}(z)$ and called the *Weierstrass $\mathcal{P}$-function.*

9. Next, you need to show that

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

where the sum is taken over all non-zero periods of $\mathcal{P}$.

Proceed as follows: first, show by norm estimates that the right-hand side converges. Next, by considering the derivative of the right hand side show that it has periods ω_1 and ω_2. Conclude that $\mathcal{P}(z)$ minus the right hand side is a constant function, which, by considering the Laurant series at zero, is actually zero.

10. Show that any even elliptic function with periods ω_1 and ω_2 can be written as

$$const \prod_{j=1}^{k} \frac{\mathcal{P}(z) - \mathcal{P}(a_j)}{\mathcal{P}(z) - \mathcal{P}(b_j)}.$$

Since $\mathcal{P}(z)$ has zero residues, its anti-derivative is a single-valued function. We normalize it so it is odd and denote $-\zeta(z)$.

11. Show that

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \neq 0} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right),$$

where the sum is taken over all non-zero periods of $\mathcal{P}$.

12. Show that $\zeta(z+\omega_1) = \zeta(z) + \eta_1$ and $\zeta(z+\omega_2) = \zeta(z) + \eta_2$ for some constants η_1 and η_2. Then show that

$$\frac{1}{2\pi i} \int_{\partial P} \zeta(z)\, dz \;=\; 1.$$

From here derive the Legendre's relation:

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i \ .$$

3. The differential Equation.

In this section you will derive a differential equation which $\mathcal{P}(z)$ satisfies. First, show that

$$\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} = -\frac{z^2}{\omega^3} - \frac{z^3}{\omega^4} - \cdots \ .$$

After summing over all periods obtain

$$\zeta(z) = \frac{1}{z} - \sum_{j=2}^{\infty} G_j z^{2j-1} \ ,$$

where

$$G_j = \sum_{\omega \neq 0} \frac{1}{\omega^{2j}} \ .$$

Since $\mathcal{P}(z)$ equals minus the derivative of $\zeta(z)$, we get:

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{j=2}^{\infty} (2j-1) G_j z^{2j-2}.$$

Now write down explicitly several initial terms of the Laurant series for $\mathcal{P}(z)$, $\mathcal{P}(z)^3$, $\mathcal{P}'(z)$, and $\mathcal{P}'(z)^2$.

Show that they satisfy

$$\mathcal{P}'(z)^2 - 4\mathcal{P}(z)^3 + 60G_2\mathcal{P}(z) = -140G_3 + \cdots \ .$$

Notice that the left-hand side is a doubly periodic function and the right-hand side has no poles thus concluding that $\mathcal{P}$ satisfies the differential equation:

$$\mathcal{P}'(z)^2 = 4\mathcal{P}(z)^3 - 60G_2\mathcal{P}(z) - 140G_3 \ .$$