Integration Workshop 2003
Project on Winding Numbers

Douglas Ulmer

Winding numbers make precise the intuitive notion of "the number of times a path goes around a point." This project uses vector calculus to set up the basics and then applies them to prove the fundamental theorem of algebra, the Brouwer fixed point theorem for a disk, and other results. Some of the basic preoccupations of algebraic topology (homotopy, integration as a pairing, degrees of mappings, ...) are met along the way.

We take a "real" point of view (so the plane is $\mathbb{R}^2$, not $\mathbb{C}$). You might find it interesting to translate everything into a more "complex" view.

1 1-forms and line integrals

1.1

Let $U \subset \mathbb{R}^2$ be an open set. A (smooth) *1-form* on U is an expression of the form $\omega = p(x, y)\, dx + q(x, y)\, dy$ where p and q are smooth (C^∞) functions on U. If f is smooth function on U, then we define df by

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy.$$

Such a 1-form is said to be *exact*.

A 1-form $\omega = p\, dx + q\, dy$ is *closed* if $d\omega = 0$ where by definition

$$d\omega = \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dx\, dy.$$

Prove that an exact 1-form is automatically closed, but that the converse is false.

1.2

A *path* $\gamma : [a, b] \to U$ is a piecewise smooth map from an interval into U. (Piecewise smooth means that γ is continuous and you can subdivide the interval into finitely many pieces so that γ is smooth on each piece.) The *endpoints* of γ are $\gamma(a)$ and $\gamma(b)$ and we say that γ is *closed* if its endpoints are equal. We write $\gamma(t) = (x(t), y(t))$ for $t \in [a, b]$.

1.3

If ω is a 1-form on U and γ is a path in U, we define

$$\int_\gamma \omega = \int_a^b (p(x(t), y(t))x'(t) + q(x(t), y(t))y'(t))\, dt$$

where the right hand side is the usual Riemann integral.

1.4

A *reparametrization* of γ is another path $\delta : [a', b'] \to U$ of the form $\delta = \gamma \circ r$ where $r : [a', b'] \to [a, b]$ is a piecewise smooth map with $r(a') = a$ and $r(b') = b$. Prove that

$$\int_\delta \omega = \int_\gamma \omega.$$

Let $-\gamma : [a, b] \to U$ denote that path with $-\gamma(t) = \gamma(a + b - t)$. (This is γ traversed backwards.) Prove that

$$\int_{-\gamma} \omega = -\int_\gamma \omega.$$

1.5

Suppose that γ and δ are closed paths in U with the same domain $[a, b]$. Then a (smooth) *homotopy* between γ and δ is a smooth function $H : [0, 1] \times [a, b] \to U$ such that $H(0, t) = \gamma(t)$, $H(1, t) = \delta(t)$,for all t and $H(s, a) = H(s, b)$ for all s. You should think of this as a family of closed paths γ_s parametrized by s such that $\gamma_0 = \gamma$ and $\gamma_1 = \delta$. We say that γ and δ are (smoothly) *homotopic*.

Prove that if γ and δ are homotopic then $\int_\gamma \omega = \int_\delta \omega$ for every closed 1-form ω on U. (Cryptic hint: Pull ω back to the square $[0, 1] \times [a, b]$ and use Green's Theorem. More details can be provided as necessary.)

1.6

Show that

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

In particular, the integral of an exact differential only depends on the endpoints of the path, not on the path itself. (Equivalently, the integral of an exact form over a closed path is zero.)

Show conversely that if a 1-form ω has the property that $\int_\gamma \omega$ only depends on the endpoints of γ, then ω is exact. (Equivalently, ω is exact if $\int_\gamma \omega = 0$ for all closed paths in U.)

1.7

Prove that the 1-form
$$\frac{-y\,dx + x\,dy}{x^2 + y^2}$$
on $\mathbb{R}^2 \setminus \{(0,0)\}$ is not exact. The traditional notation for this form is $d\theta$ which is misleading, since you just proved that there is no function θ such that $d\theta$ is the 1-form above! Explain why this notation is nevertheless appealing.

2 Winding numbers

2.1

Let $U = \mathbb{R}^2 \setminus \{(0,0)\}$ and let $\gamma : [a,b] \to U$ be a path in U. We want to think about γ in polar coordinates, but there is some ambiguity in the angle variable. I.e., if we write
$$\gamma(t) = (r(t)\cos(\theta(t)), r(t)\sin(\theta(t))), \tag{1}$$
then $\theta(t)$ is not uniquely determined. But we can make it unique by choosing a value θ_a for $\theta(a)$ and insisting that $\theta(t)$ be continuous.

Indeed, choose θ_a such that $\gamma(a) = (r\cos(\theta_a), r\sin(\theta_a))$ for some r and define $\theta(t)$ and $r(t)$ by
$$\theta(t) = \theta_a + \int_a^t \frac{-y(\tau)x'(\tau) + x(\tau)y'(\tau)}{x(\tau)^2 + y(\tau)^2}\,d\tau$$
and
$$r(t) = (x(t)^2 + y(t)^2)^{1/2}.$$

Prove that r and θ are smooth and they are the unique continuous functions making (1) true such that $\theta(a) = \theta_a$.

2.2

The quantity $\int_\gamma d\theta = \theta(b) - \theta(a)$ is called the *total angular displacement* of γ (around 0). Note that it is independent of the choice of θ_a. Prove that if γ is closed, then $\int_\gamma d\theta$ is an integer multiple of 2π. We define the winding number of γ around 0 as
$$W(\gamma, 0) = \frac{1}{2\pi}\int_\gamma d\theta.$$

Compute a few examples to make sure that this is a reasonable definition. For example, consider $\gamma(t) = (\cos(nt), \sin(nt))$.

2.3

If $p = (x_0, y_0)$ is any point in $\mathbb{R}^2$ and $\gamma : [a, b] \to U = \mathbb{R}^2 \setminus \{p\}$ is a closed path not passing through p, then we define

$$W(\gamma, p) = \frac{1}{2\pi} \int_\gamma \frac{-(y - y_0)dx + (x - x_0)dy}{(x - x_0)^2 + (y - y_0)^2}.$$

By the results in Section 1, winding numbers are invariant under reparameterization and homotopy.

2.4

The *support* of a path $\gamma : [a, b] \to U$ is by definition $\{\gamma(t) | t \in [a, b]\}$. Prove that $W(\gamma, p)$ is constant as a function of p on the connected components of $\mathbb{R}^2 \setminus \text{Support}(\gamma)$. Intuitively, we can move the point p a little without changing the winding number. Prove also that exactly one connected component of $\mathbb{R}^2 \setminus \text{Support}(\gamma)$ is unbounded, and $W(\gamma, p) = 0$ on this component.

If you know about the Jordan curve theorem, what does it say here?

2.5

Apply homotopy invariance to prove that if γ and δ are two paths in $U = \mathbb{R}^2 \setminus p$ such that for all t the line segment between $\gamma(t)$ and $\delta(t)$ does not contain p, then $W(\gamma, p) = W(\delta, p)$.

In particular, if $|\gamma(t) - \delta(t)| < |\gamma(t) - p|$ for all $t \in [a, b]$, then $W(\gamma, p) = W(\delta, p)$. This is sometimes called the "dog on a leash" theorem. Explain why.

2.6

A closed path can naturally be viewed as a piecewise smooth map from a circle to $\mathbb{R}^2$. Prove that if $\gamma : S^1 \to U = \mathbb{R}^2 \setminus \{p\}$ is a closed path and if there exists an extension of γ to the disk, i.e., a smooth function $\Gamma : D \to U$ such that Γ and γ agree on S^1, then $W(\gamma, p) = 0$. (Here S^1 is the unit circle and D is the closed unit disk.)

3 Applications

3.1

Winding numbers can be used to prove the fundamental theorem of algebra: if $f(z) = a_0 z^n + \cdots + a_n$ is a polynomial of positive degree, then $f(z)$ has a root in $\mathbb{C}$.

Clearly we may assume that f is monic (i.e., $a_0 = 1$). Suppose f has no root. Define paths $\gamma_r : S^1 \to U = \mathbb{C} \setminus \{0\}$ by setting $\gamma_r(e^{2\pi i\theta}) = f(re^{2\pi i\theta})$, i.e., we restrict f to the circle of radius r around 0 in the plane. By assumption γ_r extends to a map of the closed disk of radius r to U, so $W(\gamma_r, 0) = 0$.

On the other hand, if δ_r is the family of paths defined similarly using $g = z^n$, then $W(\delta_r, 0) = n$. Use the dog on a leash theorem to show that for large r, $W(\gamma_r, 0) = W(\delta_r, 0)$ and deduce a contradiction.

3.2

One of the most famous, and amazing, theorems of topology is the Brouwer fixed point theorem: a continuous map of the disk to itself $f : D \to D$ has a fixed point, i.e., a point $p \in D$ such that $f(p) = p$. We can prove this for smooth maps f using winding numbers. (In fact, with a little more care winding numbers can be defined for continuous maps, and then the same proof works.)

Let $\gamma : S^1 \to S^1$ be a smooth map of the unit circle to itself. We define the *degree* of γ by $\deg \gamma = W(\gamma, 0)$.

Prove that there is no retraction from D to S^1, i.e., no smooth map $F : D \to S^1$ which is the identity on S^1. (Hint: Show that $\gamma = F|_{S^1}$ would have degree 1 since it's the identity and also degree 0 because it extends to the disk.)

Now show that if $f : D \to D$ is a smooth map with no fixed points, then there exists a retraction $F : D \to S^1$. (Hint: For each $p \in D$, consider the ray from $f(p)$ through p and the point where it meets S^1.) Deduce the Brouwer fixed point theorem.

3.3

The *antipode* of a point p on a circle or sphere is the opposite point, i.e., the other point on the line through p and the center. We denote it by p^*.

Prove that if $f : S^1 \to S^1$ has the property that $f(p^*) = f(p)^*$ (i.e., it sends antipodes to antipodes), then $\deg f$ is odd. (Hint: Divide the path f in half and show that the angular displacement for each half is $2\pi(n + 1/2)$ for some integer n.)

Prove that there is no map $f : S^2 \to S^1$ such that $f(p^*) = f(p)^*$. (Hint: Cook up a map $g : D \to S^2$ such that $f \circ g$ restricted to S^1 sends antipodes to antipodes. Apply the previous paragraph to get a contradiction.)

Prove that if $f : S^2 \to \mathbb{R}^2$ is a smooth map, then there is a point $p \in S^2$ such that $f(p) = f(p^*)$. (Hint: If not, use f to construct an antipodal preserving map $S^2 \to S^1$.) This is the Borsuk-Ulam theorem and it says, for example, that at any point in time there are two antipodal points on the earth where the temperature and humidity are the same.

3.4

The Stone-Tukey (or "ham sandwich") theorem says that given three bounded regions in $\mathbb{R}^3$, there is a plane which divides each of the regions in half (in terms of volume). This is not too hard to prove from the second step in the proof of the Borsuk-Ulam theorem, or at least from the analog for continuous maps.

First convince yourself of the following 2 facts:

1. If X is a bounded region in space and L is a fixed line, then there is a unique point $P_{L,X}$ on L such that the plane perpendicular to L through $P_{L,X}$ cuts X in half.

2. If S is a sphere big enough to contain X and $Q \in S$, let $L(Q)$ be the line through Q and its antipode Q^*, and let $P_{L(Q),X}$ be the point on this line bisecting X. Then the map $Q \mapsto P_{L(Q),X}$ is continuous.

Now given a sphere S and a bounded region X as above, let $f_X : S \to \mathbb{R}$ be defined by $f_X(Q) =$ the distance from Q to the point $P_{L(Q),X}$ above. Note that f_X is continuous.

Given three bounded regions X, Y, and Z, choose a sphere S containing them all. Define a function $g : S \to \mathbb{R}^2$ by $g(Q) = (f_X(Q) - f_Y(Q), f_X(Q) - f_Z(Q))$. Argue by contradiction that there must be a point where $g(Q) = (0, 0)$ and conclude the ham sandwich theorem.