INTEGRATION WORKSHOP PROJECTS

1. Construction of Brownian motion, continuity of sample paths

The *Wiener process* $W(t)$ is a mathematical idealization of Brownian motion and is defined by the following requirements.

(1) $W(0) = 1$ almost surely.
(2) For $t > s$, $W(t) - W(s)$ is Normally distributed with variance $t - s$.
(3) If $0 < t_1 < t_2 < \ldots < t_n$, the *increments* $W(t_1)$, $W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent.

(a) Show that $E[W(t)] = 0$ and $E[W^2(t)] = t$.

The *Haar functions* h_k are defined by

$$h_0(t) = 1 \quad \text{for } 0 \le t \le 1$$

$$h_1(t) = \begin{cases} 1 & \text{for } 0 \le t \le 1/2 \\ -1 & \text{for } 1/2 < t \le 1 \end{cases}$$

$$\text{If } 2^n \le k < 2^{n+1} \qquad h_k(t) = \begin{cases} 2^{n/2} & \text{for } 2^{-n}k - 1 \le t \le 2^{-n}k + 2^{-n-1} - 1 \\ -2^{n/2} & \text{for } 2^{-n}k + 2^{-n-1} - 1 < t \le 2^{-n}k + 2^{-n} - 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Show that the Haar functions are an orthonormal family for the inner product

$$(f, g) = \int_0^1 f(x)g(x)dx$$

(c) The *Schauder functions* s_k are defined by

$$s_k(t) = \int_0^t h_k(\tau)d\tau \qquad \text{for } 0 \le t \le 1$$

Show that each s_k is non-negative and is bounded by $\min(1, \sqrt{2k^{-1}})$.

(d) ϕ is a smooth function defined on $[0, 1]$. If $(\phi, h_k) = 0$ for all k, show that $\phi \equiv 0$ on $[0, 1]$. This implies that the h_k form a *complete orthonormal basis* of $L^2([0, 1])$. (Hint: Integrate by parts for $k \ge 1$, and use the positivity of s_k)

(e) If a_k is a real valued sequence such that $|a_k| \le C_1 + C_2 k^\delta$ for some constants C_1, C_2 and for some $0 \le \delta < 1/2$, show that the series

$$\sum_{k=0}^{\infty} a_k s_k(t)$$

converges uniformly to a continuous function of $[0, 1]$.

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(f) If A_k is a sequence of independent normal variables with unit variance, show that the "Random" series $\sum_{k=0}^{\infty} A_k s_k(t)$ converges to a Wiener process $W(t)$.

This procedure is the *Levy – Ciesielski* construction of the Wiener process.

In your proof, you can use the following fact:

If A_k is a sequence of independent normal variables with unit variance, then with probability 1, there exists a K such that for all $k \geq K$, $|A_k| \leq 4\sqrt{\log k}$.

(Optional) Prove this fact. (Hint: Borel-Cantelli lemma).