Integration workshop project - Tom Kennedy
Conformal maps and SLE

1 Background

The goal of this project is to understand the Loewner differential equation and its relation to conformal maps. There is probably too much material here to go through it all during the workshop. Your goal should be to work through the proof of theorem 3. To get to it in a finite amount of time you can skip most of section 3, just reading the first definition and statement of the first theorem. If you do this you can also skip the Schwarz reflection principle in section 2.

In the last section we give a glimpse of how the Loewner equation is used to define the Schramm-Loewner evolution or SLE, and how you can numerically simulate it. This process gives random curves in the plane that describe a variety of models from probability and statistical mechanics. It was introduced by Oded Schramm in a paper that appeared in 1999. Many feel that Schramm should have won a Fields medal for this work. (The Fields medal has an age limit of 40 and Schramm was just over 40.) In 2004, a Fields medal was awarded to Wendelin Werner in large part for his joint work with Schramm and Greg Lawler on SLE.

2 Conformal maps, Riemann mapping theorem

We start by defining a conformal map.

Definition 1 *Let D and D' be open subsets of $\mathbb{R}^2$. A map $f : D \to D'$ is said to preserve angles if for every two differentiable curves γ_1 and γ_2 in D defined on the time interval $(-\epsilon, \epsilon)$ which intersect at $t = 0$, the angle formed by their tangents at $\gamma_i(0)$ is equal to the angle formed by the tangents to $f \circ \gamma$ and $f \circ \gamma'$ at $f(\gamma_i(0))$. A conformal map from D to D' is a one to one, onto, differentiable function that preserves angles.*

Exercise 1 *Let $f(x, y) = (u(x, y), v(x, y))$ be a differentiable map. Show that it preserves angles at a point if and only if its derivative (which is a 2 by 2 matrix) is equal to a positive constant times a rotation matrix, i.e., there is an $a > 0$ and a θ such that*

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = a \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{1}$$

The result of the exercise implies that the Cauchy-Riemann equations are satisfied.

Exercise 2 *Show that if the Cauchy-Riemann equations are satisfied and the derivative at the point is not zero, then there is an $a > 0$ and θ such that the above is true. So the map preserves angles.*

Thus we have shown a map is a conformal map if and only if it is a one to one, onto analytic function of D to D'. Note if f is a conformal map of D onto D', then f^{-1} is a conformal map of D' onto D.

There is a special family of conformal maps - the linear fractional transformations. They are of the form

$$f(z) = \frac{az+b}{cz+d} \tag{2}$$

where a, b, c, d are complex numbers with $ad - bc \neq 0$. Linear fractional transformations map circles onto circles if we think of lines in the plane as circles.

We will use $\mathbb{D}$ to denote the open unit disc with center at the origin and $\mathbb{H}$ to denote the upper half plane. We will use "domain" to mean an open connected subset of the complex plane. A domain is simply connected if it does not have any holes:

Definition 2 *A domain D is simply connected if the region bounded by every simple closed curve in D is contained in D, i.e., every simple closed curve in D may be continuously contracted to a point without leaving D. Equivalently, D is simply connected if $\hat{\mathbb{C}} \setminus D$ is connected.*

Recall that Cauchy's theorem says that if D is simply connected and f is analytic on D and γ is a differentiable closed curve in D, then

$$\int_\gamma f(z)dz = 0 \tag{3}$$

One of the great theorems of complex analysis is the following.

Theorem 1 *(Riemann mapping theorem) Let D be a simply connected region which is not all of $\mathbb{C}$ and let $w \in D$. Then there is a unique conformal transformation f of D onto the unit disc $\mathbb{D}$ such that $f(w) = 0$ and $f'(w) > 0$.*

Corollary: Any two simply connected domains have a conformal map between them.

A proof can be found in any first year graduate complex variables book. One of the amazing aspects of this theorem is that it does not require any smoothness of the boundary. The boundary need not even be a curve. If we want to extend the conformal map so that it maps the boundary of D onto the boundary of $\mathbb{D}$ then we need some condition on the boundary of D.

Roughly speaking, the family of conformal maps from one simply connected domain to another has three real degrees of freedom. In our statement of the theorem the constraint $f(w) = 0$ is a complex constraint and so uses two real degrees of freedom while $f'(w) > 0$ says that the imaginary part of the derivative is zero and so used the third real degree of freedom. When the map can be extended to the boundary, another common way to impose the three real constraints is to require three prescribed points on the boundary of the domain get mapped to three presecribed points on the boundary of the range.

We end this section with a useful result to extending analytic functions. If $f(z)$ is analytic, then it is easy to check that $\overline{f(\overline{z})}$ is too (defined on the obvious domain). We would like to use this fact to take an analytic function defined on a subset of the upper half plane and extend it to the refected domain in the lower half plane. Clearly $f(z)$ will need to be real on the part of the real axis in the original domain if this is to work.

Theorem 2 *(Schwarz reflection principle) Let D be a domain which is symmetric about the real axis. Let $D_+ = D \cap \mathbb{H}$. Let f be a function which is continuous on $\overline{D}_+$, analytic in D_+ and real valued on the set of reals in D. Then f may be extended to an analytic function on all of D which satisfies $f(\overline{z}) = \overline{f(z)}$.*

Exercise 3

$$f(z) = \frac{z-i}{z+i} \tag{4}$$

Show this is a conformal map of the unit disc $\mathbb{D}$ to the upper half plane $\mathbb{H}$.

Exercise 4 *If we remove the line segement from 0 to i from the upper half plane, the resulting domain is simply connected. Find a conformal conformal map from $\mathbb{H}$ minus this vertical slit onto $\mathbb{H}$. Hint: if you get stuck note that this is a special case of the next exercise.*

Exercise 5 *Let $0 < \alpha < 1$ and define*

$$f(z) = [z + 1 - \alpha]^{\alpha} [z - \alpha]^{1-\alpha} \tag{5}$$

Show that $f(z)$ conformally maps $\mathbb{H}$ onto $\mathbb{H} \setminus A$ where A is a line segment from 0 to $re^{i\alpha\pi}$ for some r. Find r as a function of α.

3 Half plane capacity

We continue to use $\mathbb{H}$ to denote the upper half plane. We do not include the real axis, so this is an open set.

Definition 3 *A bounded subset A of $\mathbb{H}$ is a "compact $\mathbb{H}$-hull" if $A = \mathbb{H} \cap \overline{A}$ and $\mathbb{H} \setminus A$ is simply connected.*

Proposition 1 *If A is a compact $\mathbb{H}$-hull, then there is a unique conformal map $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ such that*

$$\lim_{z \to \infty} [g_A(z) - z] = 0 \tag{6}$$

Proof: The Riemann mapping theorem says there exist a conformal map g of $\mathbb{H} \setminus A$ onto $\mathbb{H}$ which maps ∞ to itself. Since A is bounded it is contained in a ball $B(0,r)$ of radius r about 0 for some r. Consider $U = \{z : |z| > r\}$. On $\mathbb{H} \cap U$, g is an analytic function that can be continued to the boundary. It must map the boundary to the boundary of $\mathbb{H}$, i.e, the real axis. So by the Schwarz reflection principle, $g(z) = \overline{g(\overline{z})}$ defines an analytic continuation of g to all of U. So if we let $f(z) = 1/g(1/z)$, then f is analytic on $\{z : |z| < r\}$. $g(\infty) = \infty$ implies $f(0) = 0$. So the power series expansion of $f(z)$ about the origin is of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{7}$$

which implies g has an expansion about ∞ of the form

$$g(z) = b_{-1} z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n} \tag{8}$$

A little thought shows that since f maps parts of the real axis onto the real axis, all the b_i must be real. If we let $g_A(z) = (g(z) - b_0)/b_{-1}$, then g_A satisfies (6). We leave it to the reader to prove uniqueness. ∎

Definition 4 *Let A be a compact $\mathbb{H}$-hull. Let g_A be the unique conformal map given by the proposition. So the Laurent expansion is of the form*

$$g_A(z) = z + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \tag{9}$$

The half plane capacity, hcap(A), is b_1.

Proposition 2 *Let $A_1 \subset A_2$ be compact $\mathbb{H}$-hulls. Then $hcap(A_1) \leq hcap(A_2)$ with equality only if the sets are equal. If A is a compact $\mathbb{H}$-hull and $r > 0, x \in \mathbb{R}$ then*

$$hcap(rA) = r^2\, hcap(A), \quad hcap(A + x) = hcap(A) \tag{10}$$

Exercise 6 *Prove the above theorem.*

Exercise 7 *Find the half-plane capacity of $\mathbb{H}$ with the vertical slit from 0 to iy removed.*

Exercise 8 *Let $f(z)$ be defined as in exercise 5. Let $g(z) = f^{-1}(z)$, so g conformally maps $\mathbb{H} \setminus A$ onto $\mathbb{H}$, where A is defined as in exercise 5. Show that $g(z)$ satisfies (6). Recall that you found r as a function of α in the previous exercise. Use this to find the half-plane capacity of the segment of length 1 from 0 to $e^{i\alpha\pi}$.*

4 Loewner differential equation

Let U_t be a real valued continuous function on $[0,\infty)$. The Loewner equation is the differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z \tag{11}$$

where $\dot{}$ denotes the derivative with respect to t. The variable t is real and non-negative, while z and $g_t(z)$ are complex. The real valued function U_t is known as the "driving function." This is a differential equation in t. You can think of z as a parameter in the dif eq.

Theorem 3 *Let U_t be a continuous real-valued function on $[0,\infty)$. For $z \in \mathbb{H}$, let $g_t(z)$ be the solution of*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z \tag{12}$$

The denominator can go to zero, causing the solution to fail to exist after some finite time. Define

$$T_z = \sup\{t : g_t(z)\, exists\} \tag{13}$$

Define

$$H_t = \{z \in \mathbb{H} : T_z > t\}, \quad K_t = \{z \in \mathbb{H} : T_z \leq t\} = \mathbb{H} \setminus H_t \tag{14}$$

Then *H_t is open and K_t is closed. K_t grows with time in the sense that $s \leq t$ implies $K_s \subset K_t$. The function $g_t(z)$ is the unique conformal map of H_t onto $\mathbb{H}$ such that $g_t(z) - z \to 0$ as $z \to \infty$. Furthermore ,*

$$g_t(z) = z + \frac{2t}{z} + O(\frac{1}{|z|^2}), \quad z \to \infty \tag{15}$$

Proof: You will prove the theorem through a series of exercises. The fact that $g_t(z)$ is analytic in z and continuous in t follows from standard dif eq theorems.

Exercise 9 *Show H_t is open, K_t is closed and $s \leq t$ implies $K_s \subset K_t$.*

Exercise 10 *The goal of this exercise is to show that for each t, $g_t(z)$ is one to one. Let $z \neq w$ and define $\Delta_t = g_t(z) - g_t(w)$. Show that Δ_t satisfies*

$$\frac{d\ln(\Delta_t)}{dt} = \frac{-2}{(g_t(z) - U_t)(g_t(w) - U_t)} \tag{16}$$

Solve this dif eq with the initial condition $\Delta_0 = z - w$, and use the solution to conclude $\Delta_t \neq 0$.

Exercise 11 *This exercise shows* $g_t(H_t) \subset \mathbb{H}$. *Let* $g_t(z) = r_t(z) + iv_t(z)$ *where* r *and* v *are real valued. Show*

$$\dot{r}_t(z) = \frac{2(r_t(z) - U_t)}{(r_t(z) - U_t)^2 + v_t(z)^2} \tag{17}$$

and

$$\dot{v}_t(z) = \frac{-2v_t(z)}{(r_t(z) - U_t)^2 + v_t(z)^2} \tag{18}$$

Use these equations to show $v_t(z)$ *is decreasing when* z *is in the upper half plane and has zero derivative on the real axis. Conclude from this that if* z *is in* $\mathbb{H}$, *then* $g_t(z)$ *remains in* $\mathbb{H}$ *as long as the solution is defined.*

Exercise 12 *To show* $g_t(H_t)$ *is all of* $\mathbb{H}$ *we consider the dif. eq. run backwards in time. Fix* $t > 0$ *and define* $h_s(w)$ *by*

$$\dot{h}_s(w) = \frac{-2}{h_s(w) - U_{t-s}}, \quad h_0(w) = w \tag{19}$$

Show the solution of this dif eq exists for $s \leq t$. *Hint: what is the sign of the derivative of the imaginary part of* h_s *? Show that* $G_s = h_{t-s}$ *satisfies the original Loewner equation and has* $G_t(w) = h_0(w) = w$ *and* $G_0(z) = h_t(z)$. *So* $g_t(h_t(z)) = w$.

Exercise 13 *Verify 15 by considering the dif eq for large* z.

This completes the proof. ∎

Exercise 14 *Solve the Loewner equation with* $U_t = 0$. *Relate your answer to an exercise from section 2.*

Exercise 15 *Solve the Loewner equation for* $U_t = ct$. *You won't be able to get a completely explicit solution, but you should be able to determine the limiting behavior of the solution as as* $t \to \infty$.

Exercise 16 *You can use the conformal map in exercises 5 and 8 to find a solution to the Loewner equation which corresponds to a growing slit in the half plane. Let*

$$\phi(z) = [z + (1 - \alpha)]^\alpha [z - \alpha]^{1-\alpha} \tag{20}$$

$$f_t(z) = \sqrt{\frac{4t}{\alpha(1-\alpha)}}\,\phi\!\left(z\sqrt{\frac{\alpha(1-\alpha)}{4t}}\right) \tag{21}$$

and let $g_t = f_t^{-1}$. *Show* g_t *satisfies the Loewner equation and* K_t *is a line segment starting at* 0 *and forming an angle of* $(1 - \alpha)\pi$ *with the positive horizontal axis. Find the driving function. This is computationally heavy and it might be a good idea to use a computer to do the calculation.*

5 Schramm-Loewner evolution

Brownian motion (one-dimensional) is a real valued stochastic process on the time interval $[0, \infty)$. We denote it by W_t and take the "standard" Brownian motion which means that it has mean zero and variance t. It has the property that W_t is a continuous function on $[0, \infty)$. If you have never seen Brownian motion, think of it as some way to produce a random continuous (but not differentiable) funtion W_t on $[0, \infty)$. The Schramm-Lowener evolution is the K_t you get when you take the driving function in the Lowener equation to be $U_t = \sqrt{\kappa} W_t$ where κ is a non-negative parameter. In this case the growing set K_t will be random. It is deeply related to many things: percolation, the Ising model, the Potts model, the self-avoiding walk, the loop-erased random walk, ... It has many interesting properties and in particular has a fractal structure. For $\kappa < 4$, K_t is a curve with Hausdorff dimension $1+\kappa/8$. For $4 < \kappa$, K_t is not a curve and grows to fill the entire half plane as $t \to \infty$. Even though K_t is not a curve, it is closely related to a curve called the "generating curve." If $\kappa < 8$ this generating curve has Hausdorff dimension $1 + \kappa/8$. For $\kappa > 8$ the generating curve is space filling, i.e., it passes through every point.

Simulating SLE is a little tricky. It is defined by a dif. eq., but what we are interested in the set of points where the solution no longer exists. So to approximately find this set you have to numerically integrate the dif eq for a lot of initial conditions. One can use the dif eq to simulate SLE, but here is a more direct method.

Fix an angle $\theta \in (0, \pi/2]$ and a length $\rho > 0$. Let $f_+(z)$ be the conformal map which takes $\mathbb{H}$ onto $\mathbb{H} \setminus \{re^{i\theta} : 0 < r \leq \rho\}$, the upper half plane minus the line segment from 0 to $\rho e^{i\theta}$. This map is not unique. We make the choice unique by requiring

$$\begin{aligned} f_+(\infty) &= \infty \\ f'_+(\infty) &= 1 \\ f_+(0) &= \rho e^{i\theta} \end{aligned}$$

This normalization is slightly different from (6). Let $f_-(z)$ be the analogous conformal map for the segment from 0 to $\rho e^{i(\pi-\theta)}$. (So the range of f_- is the reflection of the range of f_+ about the vertical axis.)

Consider composing two of these maps, e.g., $f_- \circ f_+$. The effect of the second map in the composition will be to push the line segment created by the first map into the upper half plane and bend it somewhat. Because we have required that these maps send 0 to the tip of the line segment, one endpoint of the image of the first slit under the second map will be the tip of the second slit. In other words the composition $f_- \circ f_+$ will map $\mathbb{H}$ onto $\mathbb{H}$ with a curve removed. In fact we can compose multiple copies of f_- and f_+ and the resulting conformal map will send the half plane onto the half plane minus a curve.

Now we will compose a large number of copies of f_- and f_+, chosen randomly. Let X_n be a sequence of independent, identically distributed random variables with $X_n = \pm 1$ with probability 1/2. For positive integers n consider the conformal map

$$F_n = f_{X_1} \circ f_{X_2} \circ f_{X_3} \circ \cdots \circ f_{X_n} \tag{22}$$

The conformal map F_n will map $\mathbb{H}$ onto $\mathbb{H} \setminus \gamma$ where γ is a curve in the upper half plane starting at 0. To obtain SLE we let $n \to \infty$ and then let $\rho \to 0$.

Exercise 17 *Use your favorite software package to draw some pictures of SLE. See the author's home page for his pictures. What happens as you vary α? (The usual parameter κ is related to α.)*