INTEGRATION WORKSHOP PROJECTS

1. Solving elliptic PDEs

A Hilbert space H is a complete, normed linear space whose norm is induced by an inner product $\langle \cdot, \cdot \rangle$. We shall denote the norm by $\|u\| = \sqrt{\langle u, u \rangle}$.

(a) If $f : H \to \mathbb{R}$ is a continuous linear functional, show that there is a unique element $u \in H$ such that

$$f(v) = \langle u, v \rangle$$

for all $v \in H$. (This is the Riesz representation theorem for Hilbert spaces, and you don't need any measure theory to prove this special case)

A bilinear form on H is a real valued function that takes two arguments (inputs) from H and is separately linear in each argument. Clearly, the inner product is a bilinear form, but there are many other bilinear forms. A natural question is whether we can extend the above conclusion to general bilinear forms. As we see below, this idea is particularly fruitful for solving elliptic PDEs.

A bilinear form $a(\cdot, \cdot)$ on a Hilbert space H is *continuous* (or bounded) if there exists a $C < \infty$ such that

$$|a(u, v)| \leq C\|u\|\|v\|.$$

The bilinear form is *coercive* on a subspace $V \subseteq H$ if $\exists \alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|^2 \quad \forall v \in V$$

The inner product is clearly both continuous and coercive on H.

(b) If $a(\cdot, \cdot)$ is a *symmetric* bilinear form that is continuous and coercive on a closed subspace $V \subseteq H$ and $f : H \to \mathbb{R}$ is a continuous linear functional, show that, there exists a $u \in V$ (note V and not H) such that

$$f(v) = a(u, v) \quad \forall v \in V$$

(Hint: Show that $(V, a(\cdot, \cdot))$ is a Hilbert space.)

By choosing V as a *finite dimensional* subspace of a potentially infinite dimensional space H, this idea allows one to approximate the solutions of a PDE by solutions of matrix equations, *i.e.* linear algebra. This is the idea that underlies *Finite Element methods* for numerically solving PDEs on a computer.

(c) Show that for all smooth functions $g : [0, 1] \to \mathbb{R}$, the mixed boundary value problem

$$-u'' + u = g, \qquad u(0) = 0, u_x(0) = 0$$

has a unique solution.

You can use the fact that the completion of $C^\infty[0, 1]$ in the norm given by $\|v\|^2 = \int_0^1 [v^2 + (v')^2]$ is a Hilbert space, and all the elements of this space are continuous functions.

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(d) Can you prove the existence of a solution using an appropriate Fourier type basis?