THE HOPF FIBRATION

The Hopf fibration is an important object in fields of mathematics such as topology and Lie groups and has many physical applications such as rigid body mechanics and magnetic monopoles. This project will introduce the Hopf fibration from the points of view of the quaternions and of the complex numbers.

Consider the standard unit sphere $S^n \subset \mathbb{R}^{n+1}$ to be the set of points $(x_0, x_1, \ldots, x_n)$ that satisfy the equation

$$x_0^2 + x_1^2 + \cdots + x_n^2 = 1.$$

One way to define the Hopf fibration is via the mapping $h : S^3 \to S^2$ given by

$$h(a, b, c, d) = (a^2 + b^2 - c^2 - d^2, 2(ad + bc), 2(bd - ac)). \tag{1}$$

You should check that this is indeed a map from S^3 to S^2.

(1) First, we will use the *quaternions* to study rotations in $\mathbb{R}^3$. As a set and as a vector space, the set of quaternions is identical to $\mathbb{R}^4$. There are 3 distinguished coordinate vectors–$(0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$–which are given the names i, j, k respectively. We write the vector (a, b, c, d) as $a + bi + cj + dk$.

The *multiplication rules* for quaternions can be summarized via the following:

$$i^2 = j^2 = k^2 = -1,$$

$$ij = k,\ jk = i,\ ki = j.$$

Is quaternion multiplication commutative? Is it associative?

We can define several other notions associated with quaternions. The *conjugate* of a quaternion $r = a + bi + cj + dk$ is $\bar{r} = a - bi - cj - dk$. The *norm* of r is $||r|| = \sqrt{a^2 + b^2 + c^2 + d^2}$. Finally, the *multiplicative inverse*, r^{-1}, of a non-zero r is $r^{-1} = \frac{\bar{r}}{||r||^2}$.

(2) We can use a quaternion r to determine a mapping, $R_r : \mathbb{R}^3 \to \mathbb{R}^3$. To a point $p = (x, y, z) \in \mathbb{R}^3$, we associate a *pure* quaternion, $p = xi + yj + zk$. The product rpr^{-1} is also pure (check this), and so it can be thought of as a point in $\mathbb{R}^3$, (x', y', z'). Define

$$R_r(x, y, z) = (x', y', z').$$

- Show R_r is a linear mapping.
- Let $r = a + bi + cj + dk$ be a unit quaternion. If $r = \pm 1$, show that R_r is the identity map. If not, show that R_r is the rotation about the axis vector (b, c, d) by the angle $\theta = 2\cos^{-1}(a) = 2\sin^{-1}(\sqrt{b^2 + c^2 + d^2})$. (Hint: show that R_r is an isometry and that (b, c, d) is an eigenvector.)

- Let r and s be unit quaternions. Show that

$$R_r \circ R_s = R_{rs}.$$

- The set of rotations in $\mathbb{R}^3$ with the operation of composition is a group known as $SO(3)$. S^3 with the operation of quaternion multiplication is also a group. Verify that

$$S^3/\{1,-1\} \cong SO(3).$$

(3) Let $P_0 \in S^2$ be $(1,0,0)$. The Hopf fibration is defined by

$$r \mapsto R_r(P_0).$$

Check that this agrees with Eq. 1.

- What is the fiber of the Hopf map over the point $(1,0,0)$? What shape does it have?
- Show that $h^{-1}((-1,0,0)) = \{ke^{it}\}_{0 \leq t \leq 2\pi}$.
- Let $P = (p_1, p_2, p_3) \in S^2$, and define r_P to be

$$r_P = \frac{1}{\sqrt{2(1+p_1)}}((1+p_1)i + p_2 j + p_3 k).$$

For P not equal to $(1,0,0)$ or $(-1,0,0)$, show that the fiber over P can be written as $\{r_P e^{it}\}_{0 \leq t \leq 2\pi}$.

(4) We can use stereographic projection to "see" inside S^3 and view pictures of the fibers of the Hopf map. Recall that stereographic projection is a map $s : S^3 \setminus (1,0,0,0) \to \mathbb{R}^3$ given by

$$s(w,x,y,z) = (\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w}).$$

- What is $s \circ h^{-1}((1,0,0))$? $s \circ h^{-1}((-1,0,0))$?
- For an arbitrary point $P = (p_1, p_2, p_3)$ not equal to $(1,0,0)$ or $(-1,0,0)$, show that $s \circ h^{-1}(P)$ is a circle that intersects the yz-plane in exactly two points and that this circle is linked with $s \circ h^{-1}((-1,0,0))$. Show that the x-axis passes through the interior of $s \circ h^{-1}(P)$.
- Show that the images under s of any two fiber circles are linked.
- Use computer software (such as Mathematica) to view some of the fibers of the Hopf map.

(5) If one thinks of S^3 as the unit sphere of $\mathbb{C}^2$ and of S^2 as $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then we can define the Hopf map to be $h(z_1, z_2) = \frac{z_1}{z_2}$.

- Let $D^2 = \{u \in \mathbb{C} : |u| \leq 1\}$ and $S^1 = \partial D^2$. Show that $\phi_1 : D^2 \times S^1 \to h^{-1}(D^2)$ given by $\phi(u, z) = \frac{1}{\sqrt{1+|u|^2}}(uz, z)$ is a fiber preserving homeomorphism. Similarly, show that $\phi_2 : D^2 \times S^1 \to h^{-1}(\hat{\mathbb{C}} \setminus intD^2)$ given by $\phi_2(u, z) = \frac{1}{\sqrt{1+|u|^2}}(z, \bar{u}z)$ is also a fiber preserving homeomorphism. What can you conclude about the Hopf map?
- Given an arbitrary point on a torus, four circles can be drawn through it–the circle in the plane parallel to the equatorial plane of the torus, the circle in the plane perpendicular to this, and two others. These two other circles are called *Villarceau circles*.
  Given a circle of latitude, C, on S^2, we know what the stereographic projection of $h^{-1}(C)$ should be (a torus). Show that, under stereographic projection, the fibers of points on C are linked Villarceau circles. You might also use computer software to visualize these circles on the torus.